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THE MATHEMATICAL ASSOCIATION

AN ASSOCIATION OF TEACHERS AND STUDENTS
OF ELEMENTARY MATHEMATICS

*

'I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto.'

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Dr. E. A. MAXWELL
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PASTORS AND MASTERS

Presidential Address to The Mathematical Association, April 1961

By E. A. MAXWELL

I begin by expressing my sincere thanks to the Association for the great honour bestowed on me in my election to the office of President. I realised when elected that the honour was indeed great; but it has also happened that, in the course of the year, I have had occasion to look through the list of those who have held the office before me, and to find one's name in that company is both humbling and—let me be honest—exhilarating. In particular, and this will be relevant for what I have to say later, the records speak not only of men and women famous throughout the scientific world, but also of those whose special genius has led them to the top of the teaching profession, where their memory remains alive to generations of pupils of of whom, doubtless, many are here today. It is a source of real strength to this Association that it honours not only those who create but also those who pass on the fruits of creative labour, sometimes even to those who will themselves become the creators in their turn.

It has been my privilege and pleasure to speak at other times before the Association or its Branches, and on those occasions I have tended to deal somewhat light-heartedly with topics of greater or lesser importance. There are, however, many very serious matters which claim our attention at this time, and it is with them that I have judged it more appropriate to occupy this address. Much of what I shall say is not concerned directly with the subject of mathematics, but I hope you will not think it irrelevant for all that. Some of it, indeed, is addressed beyond the teaching profession to what I shall call the non-teaching *'laity'*; not many of them may come across my

words, but I should like to record them nevertheless, in the hope that perhaps a little may penetrate outside.

In most parts of the world today there is a serious shortage of teachers, as those of you who heard Mr. Langford's brilliant analysis in his Presidential Address will have realised with special immediacy. This shortage is particularly acute in mathematics, and urgent steps are necessary if the shortage is not to become catastrophic. For example, at a recent examination at top-school level, I marked the script of a girl who was obviously having difficulty. At the end she wrote some words that everyone concerned about the supply of teachers might take as a text (my quotation is from memory, but nearly exact):

> "I am sorry I have not done better, but all the teaching I have been able to get was 5 hours of discussion. I mention this not as an excuse, but as a reason."

The position could not be stated more cogently, nor with greater authority.

On the other side, though, it would be quite unfair not to record at this stage that a great deal is even now being done to relieve the situation, and many people of real influence are deeply concerned. The fact that so much nevertheless remains is a measure of the problem.

I have given to this Address the title, "Pastors and Masters" for reasons that are probably obvious to begin with and that will, in any case, become clearer later. I approach the theme by recalling the familiar concept that any proper consideration of the problems of education must be concerned with the whole man and not only with isolated aspects of him. Inevitably, what I say in this context must be coloured by the fact that I myself have grown up in the Christian tradition, which is bound to affect all my ways of thinking; but I believe that what I have to say is unlikely to cause any deep disagreement from those who claim other allegiances.

The Church, for many centuries, has taught that the whole man is a unit, formed by bringing together three characteristic aspects—body, mind and spirit. I should not like to have to define any of these terms (even body) too precisely, but I am sure that, in broad outline, we are all aware of what is meant. The unfortunate thing, from the teacher's point of view, is that these three aspects cannot be kept separate and treated in isolation, but that each one of these keeps impinging on the others. A headache is a disturbing background both to the binomial theorem and to the shorter catechism—even when the latter, to suit modern tastes, has had the devil taken out of it.

The problem of the teacher is a complex one, for his raw material consists of human beings who, while under him, are greatly underdeveloped in all three elements: body, mind and spirit; and the

three faculties are then (as, indeed, later) very much at war with each other—with body as an easy victor. It is, I imagine, impossible that any one teacher should tackle all three aspects himself; but the true teacher must be aware of the whole development of his pupils and sensitive to everything that is going on. This means, I feel sure, that the task of the teacher is much more complicated than is often realised by people who see the profession only from the outside. Of course, there are bad teachers and good teachers, just as there are, say, bad mechanics and good mechanics; but most teachers, like most mechanics, take their work much more seriously than they are always given credit for, and it is certainly true that most teachers are keenly interested in the whole growth of their pupils.

I come now to a parallel argument which I should like to elaborate at a more general level. The affairs of this world are, in a broad sense, also directed towards precisely those three aspects of body, mind and spirit to which I have referred. These are, on the whole, the goals towards which we apply our efforts; and I propose to consider briefly the weight that the community seems to place upon them when that weight is expressed in terms of monetary reward. I should emphasise at once that I am not concerned with any of the particular and pressing problems of salary revision that are so much before us at present; that would not seem to me to be proper to the present occasion. What I am concerned about is to assess the relative importance which the community seems to attach to body, mind and spirit in terms of its willingness to recompense those who deal with them.

A short time ago the affairs of the body were summarised in the familiar phrase, "Never had it so good"; and though that phrase does seem to have lost a little of its splendour at the moment, it is nevertheless true that, in purely physical matters, we as a nation are extremely well off. This is not to say that everyone has easy access to even all those things that may be judged necessities; but compared with, say, thirty years ago, or with the state of affairs in some other countries, the progress has been remarkable. The body has indeed flourished, and I think it is quite certain that those who provide for our physical needs and comforts now receive much better reward in money and in kind for their labours. This is as it should be, and we are all happy to see it, without in any way committing ourselves as to whether a time will come when that level might not be even higher.

The affairs of the mind, however, present a very different picture. The people of our nation are indeed prepared to pay, even though there is a little grumbling at times, for their television sets and their motor cars, but they seem less willing to respond equally towards those whose wares are the more intangible things of the mind. We

must admit, of course, that this is a not unnatural state of affairs: you can go into a shop and put down £25 for a gramophone by direct and immediate purchase, whereas nobody even expects a child to come forward for a shilling's worth of quadratic equations. The link is more remote. That is why those responsible for the welfare of the teaching profession must be continually alert, so that the value of the wares of the mind does not become debased.

To complete my survey, I add a little on behalf of those who deal with the even more intangible wares of the spirit. Difficult though things are for many members of the teaching profession, they approach real and active hardship for the clergy, many of whom are being forced into standards which everyone here would deplore. I am aware that I am moving beyond what is relevant to this particular meeting, but I include it in my discussion so that you may register the fact and also so that I may complete my analysis.

The upshot of what I have been saying is this: that twentieth-century man, with all his progress and technical achievements, has reached an attitude that will give high reward to those who look after his physical well-being, much more grudging reward to those whose concern is his mind, and, to be honest, disgracefully low reward to those who care for his spirit. I am not here concerned to know what the correct proportions should be—if, indeed, the word 'correct' can have any meaning in this context; what I am concerned to establish is that the community as a whole should look anxiously towards its sense of values.

There is, of course, another side to all this which we should be wrong to ignore. It might be definitely harmful if the community went so far to the other extreme that its scales of financial reward were geared to ultimate benefit bestowed, even if that were in any way possible. The pastor and master have a satisfaction in their work that may be denied to many who earn larger salaries in industry or commerce. The real teacher just cannot keep his fingers off the chalk and duster, and he has his thrills from year to year as he watches the growth and development of his pupils. He knows as he enters the profession that he could almost certainly receive better pay for comparable ability elsewhere, yet it is teaching that he wants and teaching that he does. But that is no reason why the community should not face its own responsibilities and insist that its pastors and masters are properly looked after.

It happens that I, who am neither school-teacher nor parson, have become closely linked with many matters concerning these two great professions, where I have many friends. It would give me the greatest pleasure if someone in authority and with a voice more public than mine were to come across these ideas and crusade for a return to a proper sense of values in our national life. The

call must go out to the laity—to those, that is, who themselves are neither pastors nor masters, but who care for the things of the mind and the spirit no less than for those of the body. The public at large, in fact, must learn to see its responsibilities and then to act according to what it sees.

As I warned you at the beginning, what I have been saying so far is not of direct concern to teachers of mathematics as such, though it is a necessary background, at present, to all our thinking. I should now like to look a little more closely at our own problems.

I hope that it is clear, both from what I have just been saying and also from my personal contacts with many of you, that I have the highest regards for the teaching profession as a whole and a real sympathy with many of its problems. I confess, though, that I feel occasional doubts when I read remarks about the "professional status" of teachers coupled with an implication that what is basically required to raise their standing is an increase in paper qualifications. That sound learning and training are essential, I would not for a moment deny; but I do not believe that the laity, to whom I hope to appeal through this address and of whom I myself am a member, give such matters the thought of a single moment. In the words of Burns, "The man's the man for a' that," and the real teachers, by whom I mean the vast majority, already have ample status in the regard of hundreds of pupils each, who remain forever indebted to them. Those who do not capture that regard will never be granted the reality of the status, whatever their apparent qualifications may be. The public is unlikely to be awakened or won by such considerations.

I emphasise that point now, for I think that it is precisely here that Associations such as ours can be of tremendous value. To take our own case, the Mathematical Association has a record of growth of which it has every reason to be proud. It has been my privilege to see for many years the work undertaken on our behalf by those who might very nearly be called permanent officials in spite of annual re-election, and I have been continually impressed by their soundness of judgment and devotion to our interests. We saw at our Business Meeting a short time ago that their efforts have, indeed, been so successful that a reorganisation and expansion have proved necessary. I wonder whether I might take the opportunity, too, as I am sure our Officers would wish, of paying tribute to the silent supporting work of Mr. Gundry who deals so ably with the formidable business of our office in Gordon Square. The problem I feel moved to put before the Association now is whether we of the rank and file—the President, you will remember, is an Officer for one year only—have been giving them all the support of which we are capable so that our Association can achieve the whole of its potential impact.

I have been avoiding so-called statistics resolutely, despite many temptations, in the course of this address, but it is obvious, even without figures, that there remain many teachers and others closely concerned with mathematics who have not yet felt the urge to join us. Some, of course, just do not like this kind of thing; but they are in the minority, and I feel sure that there are many others ready to join us, given the proper encouragement. As we heard a short time ago, our growth is becoming remarkable, but we ourselves could make it even better. I believe that an Association like ours grows best by personal contacts, and each one of us can give valuable help by letting our colleagues know about it and persuading them to come in.

The advantages of a strong and widely representative Association such as we are becoming and, indeed, have become cannot be overestimated. A teacher in isolation may encounter endless difficulties, to which even discussion with immediate colleagues may not supply an adequate answer. Here, we are gathered together from many places and from many types of schools. I had necessarily to prepare these remarks before coming to the meeting, but I was delighted to learn from the Local Committee that the numbers expected to attend had forced an expansion in the accommodation required. It is perhaps as well that those numbers should be measured in hundreds rather than in thousands; but, all the same, much that we can offer is missed by those who do not sit with us at lunch or dinner—or even breakfast—casually drifting into discussions that no right-minded Programme Committee could ever set down for general debate. The meetings of our Branches are invaluable for this side of our work; I have attended meetings of most of them now, and have been most impressed by the keenness and friendliness to be found there.

It may not be out of place to pause for a moment just to look at the things that the Association offers. They were referred to in the Annual Report of Council, but the attempt to place all our efforts in a single perspective may prove illuminating.

The Gazette is, of course, the great link to bind us all together, and I for one always find particular interest in the short notes dealing with new aspects or extensions of familiar work, or with fresh ideas for teaching it. I am especially glad when those notes come from people whom I have not yet met, for I feel then that new voices are beginning to make themselves heard. This is, in any case, an outstanding opportunity open to all of us to pool ideas and also to recover that freshening of the mind which such writing always brings. The fact that the teacher is occupying himself with these ideas cannot fail to "carry across" to his pupils in the class-room.

The Reports of the Association are well known, and there is no point in saying much about them here. I often think, though, that

the people who profit most by them are, in the nature of the case, those who spend long hours discussing them. (In parenthesis, I am astonished every now and again when I recall the number of firm friendships that I have made on the sole basis of mutual criticism of the drafts of mathematical reports and examination questions.) My plea at this point is that the number of people engaged in our actual activities should be as large as possible. The introduction of "fresh blood" is not easy, and young new members have a natural—and wholly admirable—reluctance against pushing forward. Perhaps the informal lunch-and dinner-time talking, to which I have already referred, is typical of the ways in which we may get to know each other.

Passing quickly, but very appreciatively, over the valuable work done by our Library and Problems Bureau, I feel that, in this brief review, I ought to say a few words about our new Diploma. It is always unwise to comment on an event that has not fully happened, but it seems already certain that the Diploma will have a massive impact on the teaching of mathematics. The rules and specimen papers, as you know, are now public property, and a throng of candidates is waiting impatiently for the privilege of being first to sit the examination. The work involved in making the preparations has been enormous, and I can assure you from a privileged position of gentle activity inside, that the Association owes a deep debt of gratitude for all that has been accomplished by those who have worked so hard for us.

You will know, too, that many activities of general mathematical interest are taking place with which the Association is not directly involved. Members of the Association sit, in private capacity, of course, on most of the Committees concerned with school mathematics at all levels; and, again, representatives are even now preparing reports for submission to the Congress of Mathematicians to be held at Stockholm in 1962.

The upshot of this, and much more, is a record of activity that surely must make a deep impression on the teaching of our subject in all types of schools and greatly enhance the status of those taking part.

It does, however, seem to me to present a problem that the Association must tackle firmly in the not very distant future. If I am right, we shall have to be thinking soon about the whole problem of our accommodation, which is bound to become more and more cramped as our activities develop. Council is well aware of the problem, but we should all be concerned for our own premises. This leads to a corollary: our finances have been wisely and firmly guided by a succession of Treasurers, and we have every cause for satisfaction, as we have just heard. But development must be expensive, and it may

please you to hear a President declare with a firm Scottish accent that the power and influence of the Association might be increased beyond recognition if we were to take the plunge and increase our subscription right beyond our immediate known needs. Oddly enough, the effect of a high subscription (if not too high) is often to stimulate interest, and so to increase membership, among those who sense that it must be a symptom of something happening. All this, of course, is purely personal opinion, and may be wrong, but I hope you may find it worthy of discussion among yourselves some time.

If I have succeeded in making myself clear, the whole of this talk links itself together towards a single theme—the worthwhile-ness of teaching and the proper status of the profession. I have suggested that those of us who belong to what I have called the 'laity' should reform our thinking to give proper attention to the affairs of the mind and the spirit, and I have shown how Associations such as ours can, through their manifold activities, raise the whole standard of teaching in their subjects and therefore serve to inform the 'laity' about what is already being done. I have paid tribute to those who have guided the affairs of our own Association so wisely, and have suggested that we, of the rank and file, should back them up vigorously so that the Association may set its sights higher and higher and grow, as it can, to achieve an influence that will ensure for the teaching of British mathematics a place even more secure than that which it already holds. This is both short-term and long-term policy, but so much has been accomplished that we have ready to hand a foundation on which, I believe, a magnificent structure can be erected.

The end of this address has been closely concerned with ourselves, our own Association and our own attitude towards it. For this I make no apology. The crisis in the teaching of mathematics is now upon us, heavily but, as yet, by no means overwhelmingly; and it is to the members of the Mathematical Association that the public ought to look for guidance. A progressive, virile and happy body of teachers encourages others to join, and I firmly believe that, in the last resort, there is no other way.

E. A. M.

Queens' College, Cambridge

NEWTON'S DISCOVERY OF THE GENERAL BINOMIAL THEOREM

By D. T. Whiteside

Newton was the greatest mathematician of the seventeenth century. Today, almost three centuries afterwards, we are just beginning to realize the full extent and variety of his achievement. Much of his mathematical work has never been published (though it ranges far through the fields of projective geometry and general point-correspondences to number theory and an exhaustive treatment of interpolation by finite differences) and is now, with few exceptions, to be found only in little known manuscripts in the Cambridge University Library. But Newton is remembered above all, and following his own wish, for his creation of the fluxional calculus and the theory of infinite series, two strands of mathematical technique which he bound inseparably together in his "analytick" method. Of this method the binomial expansion, $(1 + a)^n = 1 + \binom{n}{1}a + \binom{n}{2}a^2 + \ldots$ ($|a| < 1$, n real), is a keystone, and its general formulation was a highlight of the magical year 1665 when he was in the prime of his age for invention.

What led Newton to his discovery, and what was the sequence of his thought? Henry Briggs in the 1620's knew the particular case $n = \frac{1}{2}$ of the expansion,[1] but in all the tens of thousands of sheets of Newton's manuscripts which still survive there is nothing which suggests any kind of link between the two. However Newton himself was only too ready to sketch his ideas as they grew out of his reading of Wallis' "arithmetica infinitorum"—to Leibniz in 1676, to Wallis himself a little later and finally, as an old man at the time of the priority dispute with Leibniz, to anyone ready to listen to his case.[2] Reconstruction of the process of Newton's thought as it can be restored from such varied hints has been a favoured pursuit of recent historians[3] and it is interesting to compare their reasoned guesses with the extant manuscripts (CUL. Add 3958·3:70–72 and Add 4000:18–19v) in which Newton wrote up his findings and from which he quoted in many later accounts of his discovery.

This is not the place to evaluate in detail Newton's debt to Wallis or to point Newton's superiority in his widened concept of interpolation over integral sequences—that would involve, among other things, a deep consideration both of the inadequacies of Wallis' viewpoint and of the growth of the concept of free variable in the 17th century. We can say, however, that the curious tabular form in which Newton develops his results, while it is explicitly an extension of the "Pascal" triangle of binomial coefficients widely known

at the time, is taken straight from the later propositions of "arithmetica infinitorum" in which Wallis interpolates his '$\Box$' $(= 4/\pi)$ or $f(\tfrac{1}{2}, \tfrac{1}{2})$ as the continued product $\tfrac{2}{1} \times \tfrac{2}{3} \times \tfrac{4}{3} \times \tfrac{4}{5} \times \tfrac{6}{5} \times \tfrac{6}{7} \times \ldots$ by tabulating, in an analogous array, a uniform set of values of the integral function $f(r, s) = \dfrac{1}{\int_0^1 (1 - x^{\frac{1}{r}})^s \,.\, dx} = \dfrac{\Gamma(r + s + 1)}{\Gamma(r + 1) \times \Gamma(s + 1)}$ $\left[= \binom{r+s}{r} \text{ for } r, s \text{ positive integers}\right]$ and deriving therefrom by inspection certain (provable) inequalities. We find too in Newton's treatment, inserted to keep the general pattern of the argument, such Wallisian idiosyncracies as the use of "$\dfrac{0}{0}$" for unity. But let Newton take up the story.[4]

"In the winter between the years 1664 & 1665 upon reading Dr. Wallis's Arithmetica Infinitorum & trying to interpole his progressions for squaring the circle, I found out first an infinite series for squaring the circle & then another infinite series for squaring the Hyperbola & soon after." This is exactly the sequence of invention which we find in the earlier of the MSS, Add 3958·3:72 and Add 4000:18–19v, where he interpolates in the sequences (n variable) $\int_0^x (a^2 - x^2)^n \,.\, dx$ and $\int_0^x (a^2 + x^2)^n \,.\, dx$, $\int_0^x a^2(b + x)^n \,.\, dx$ the values $n = \tfrac{1}{2}$ and $n = \tfrac{1}{2}, -1$ to derive respectively the areas under the circle $y^2 = a^2 - x^2$ and the hyperbolas $y^2 = a^2 + x^2$, $y(b + x) = a^2$. Apparently almost at once he polished his first awkward if suggestive writings into the firmer and logically more finished version of Add 3958·3:70–71 (which is set out in four propositions with the suggestion of a fifth). For simplicity of treatment the original sequence of invention is reversed and Newton begins with the problem of interpolating $\int_0^x a^2(b + x)^{-1} \,.\, dx$ in the sequence $\int_0^x a^2(b + x)^n \,.\, dx$, which he now reduces to the problem of interpolating $a^2(b + x)^{-1}$ in the binomial sequence $a^2(b + x)^n$ (from which the result follows by simple integration). Clearly we can, for n positive integral, evaluate the coefficients and set them up in a Pascal triangle. To fix his ideas Newton extends the table by adding zeros to fill in the gaps and writes down:

$$\text{“}\ \frac{aa}{b + x} \quad .\,1 \times aa \quad .\,1 \times aab \quad .\,1 \times aabb \quad .\,1 \times aab^3 \quad \ldots$$

$$0 \times \frac{aax}{b} \quad .\,1 \times aax \quad .\,2 \times aabx \quad .\,3 \times aab^2x \quad \ldots$$

$$0 \times \frac{aaxx}{bb} \,.\, 0 \times \frac{aax^2}{b} \,.\, 1 \times aaxx \,.\, 3 \times aabx^2 \;\ldots$$

$$0 \times \frac{aax^3}{b^3} \,.\, 0 \times \frac{aax^3}{b^2} \,.\, 0 \times \frac{aax^3}{b} \,.\, 1 \times aax^3 \quad \ldots$$

$$0 \times \frac{aax^4}{b^4} \,.\, 0 \times \frac{aax^4}{b^3} \,.\, 0 \times \frac{aax^4}{b^2} \,.\, 0 \times \frac{aax^4}{b} \quad \ldots$$

$$\ldots \quad . \quad \ldots \quad . \quad \ldots \quad . \quad \ldots \qquad \ldots\text{''}$$

"Now to reduce y^e first terme $\frac{aa}{b+x}$ to y^e same forme w^{th} y^e rest, I consider in what progressions y^e numbers prefixed to these termes proceede, & find y^m to be such y^t any number added to y^e number above it is equall to y^e number following it. Whence any termes may be found w^{ch} are wanting, as in y^e annexed Table." Applying this rule Newton expands the Pascal triangle backwards and finds that a not dissimilar pattern runs through the negative half of the array of binomial coefficients. Thus:

...	1	1	1	1	1	1	1	1	1	1	1	...
...	−4	−3	−2	−1	0	1	2	3	4	5	6	7	...
...	10	6	3	1	0	0	1	3	6	10	15	21	...
...	−20	−10	−4	−1	0	0	0	1	4	10	20	35	...
...	35	15	5	1	0	0	0	0	1	5	15	35	...
...	−56	−21	−6	−1	0	0	0	0	0	1	6	21	...
...	84	28	7	1	0	0	0	0	0	0	1	7	...
...	.	.	.	.	.	.	.	.	.	.	.	.	...

"Also any terme to w^{ch} these numbers are prefixed, being multiplyed by b, produceth y^e following litterall terme. Or y^e higher terme multiplyed by $\frac{x}{b}$ produceth y^e lower terme . . . Whence it appears y^t $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{bb} + \frac{aaxx}{b^3} - \frac{aax^3}{b^4}$ &c." This is, of course, the binomial expansion

$$\frac{a^2}{b}\left(1 + \frac{x}{b}\right)^{-1} = \frac{a^2}{b}\left(1 - \frac{x}{b} + \frac{x^2}{b^2} - \ldots\ldots\right),$$

but we note that the expansion is not developed by algebraic division: as Newton was on several occasions at pains to point out, such division (as algebraic root-extraction for the case of $(a^2 \pm x^2)^{\frac{1}{2}}$ below) was used only to check the series-expansion and played no part in its discovery.

Newton had been successful in extending the Pascal triangle backwards to include the negative part of the array. But quadrature

of the general circle segment by $\int_0^x (a^2 - x^2)^{\frac{1}{2}} \,.\, dx$ requires that we interpolate fractional values between the existing ones in this array. How are we to do this? Once more Newton tries to find some skein of the pattern which runs through the table. The general solution which he found is sketched in his proposition 4 (70v–71): "To find two or three intermediate termes in y^e above mentioned table of numerall progressions, I observe y^t those progressions are of this nature, viz.

$$\begin{array}{lllll}
\ldots.\,[a] & .\,a & .\,a & .\,a \\
\ldots.\,[b-c] & .\,b & .\,b+c & .\,b+2c \\
\ldots.\,[d-e+f\,] & .\,d & .\,d+e & .\,d+2e+f \\
\ldots.\;\ldots\;\ldots\;\ldots & .\,g & .\,g+h & .\,g+2h+i \\
\ldots.\;\ldots\;\ldots\;\ldots & .\,l & .\,l+m & .\,l+2m+n \\
\ldots.\;\ldots\;\ldots\;\ldots & .\,r & .\,r+s & .\,r+2s+t \\
 & .\,. & .\,. & .\,.
\end{array}$$

$$\begin{array}{lll}
.\,a & .\,a & .\,\ldots \\
.\,b+3c & .\,b+4c & .\,\ldots \\
.\,d+3e+3f & .\,d+4e+6f & .\,\ldots \\
.\,g+3h+3i+k & .\,g+4h+6i+4k & .\,\ldots \\
.\,l+3m+3n+p & .\,l+4m+6n+4p+q & .\,\ldots \\
.\,r+3s+3t+v & .\,r+4s+6t+4v+w & .\,\ldots \\
\quad .\,. & \quad .\,. & \ldots
\end{array}$$

And y^t y^e summe of any terme & y^e terme above it is equall to y^e terme following it at the distance of y^e termes in y^e s[ai]d numerall table." To show how this empirical observation may be applied he continues: "Suppose I would find y^e meane termes in y^e 3^d progression,

$$\begin{array}{llll}
\ldots\quad 3 & .\quad x & .\quad 1 & .\quad x \\
\ldots\, d-4e+10f & .\,d-3e+6f & .\,d-2e+3f & .\,d-e+f \\
 & \quad x & .\quad 1 & .\quad x \\
 & d+3e+3f & .\,d+4e+6f & .\,d+5e+10f
\end{array}$$

$$\begin{array}{lll}
.\,0\,.\quad x & .\quad 0 & \\
.\,d\,.\,d+e & .\,d+2e+f & . \\
.\quad 3 & . & \ldots \\
.\,d+6e+15f & . & \ldots\text{''}
\end{array}$$

Here Newton equates "y^e termes of y^t progression & of y^e correspondent litterall progression," and clearly if we can find values of d, e and f which satisfy the infinite set of simultaneous equations (assumed consistent)

$$\begin{array}{ll} \ldots\ldots\ldots\ldots\ldots\ldots & = \ldots \\ d - 4e + 10f & = 3 \\ d - 2e + 3f & = 1 \\ d & = 0 \\ d + 2e + f & = 0 \\ d + 4e + 6f & = 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots & = \ldots \end{array}$$

we can interpolate the arithmetical means required. Resolving, $d = 0$, $e = -\frac{1}{8}$ and $f = \frac{1}{4}$, so that "y^e progression must be

$$\ldots 3 \,.\, \tfrac{15}{8} \,.\, 1 \,.\, \tfrac{3}{8} \,.\, 0 \,.\, -\tfrac{1}{8} \,.\, 0 \,.\, \tfrac{3}{8} \,.\, 1 \,.\, \tfrac{15}{8} \,.\, 3 \ldots\text{''}.$$

More generally, Newton gives the central terms of the Pascal triangle expanded to include $\frac{1}{2}$-intervals, derived by an analogous procedure:

....	1	1	1	1	1	1	1	1	1	1	1	....
....	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	....
....	3	$\frac{15}{8}$	1	$\frac{3}{8}$	0	$-\frac{1}{8}$	0	$\frac{3}{8}$	1	$\frac{15}{8}$	3	....
....	-4	$-\frac{35}{16}$	-1	$-\frac{5}{16}$	0	$\frac{1}{16}$	0	$-\frac{1}{16}$	0	$\frac{5}{16}$	1	....
....	5	$\frac{315}{128}$	1	$\frac{35}{128}$	0	$-\frac{5}{128}$	0	$\frac{3}{128}$	0	$-\frac{5}{128}$	0	....
....	-6	$-\frac{693}{256}$	-1	$-\frac{63}{256}$	0	$\frac{7}{256}$	0	$-\frac{3}{256}$	0	$\frac{3}{256}$	0	....

Similar considerations allow the interpolation of the basic Pascal triangle at $1/p$ intervals, p integral (and he gives the table for $p = 3$).

Clearly Newton has his circle quadrature, and the binomial expansion for $n = \frac{1}{2}$ in particular, by using the coefficients in the 6th column of the table above—specifically

$$\int_0^x (a^2 - x^2)^{\frac{1}{2}} \,.\, dx \left[= \int_0^x a\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} . \, dx \right]$$

$$= \int_0^x a\left(1 + \frac{1}{2}\left(-\frac{x^2}{a^2}\right) - \frac{1}{8}\left(-\frac{x^2}{a^2}\right)^2 + \frac{1}{16}\left(-\frac{x^2}{a^2}\right)^3 - \ldots\right)dx.$$

It only remained to give a general rule for the various coefficients without having to perform for each special case a laborious interpolation on the above lines, that is, the formation rule for the general binomial coefficient $\left(\begin{matrix} x/y \\ k \end{matrix}\right)$: this Newton sets out (on f 71) in all its generality, if a little cumbrously to the modern eye, as

$$\text{“}\frac{1 \times x \times \overline{x - y} \times \overline{x - 2y} \times \overline{x - 3y} \times \overline{x - 4y} \times \overline{x - 5y} \times \overline{x - 6y}}{1 \times y \times 2y \times 3y \times 4y \times 5y \times 6y \times 7y}\ \&\text{c.”}$$

Newton had all a young man's intoxication with his discovery, and he spent much of the summer of 1665 in using the "Mercator" series

$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ (which is an immediate corollary of $\int_0^x (1 + x)^{-1}\, dx$ when expanded by the binomial theorem) in lengthy but fascinating calculation of many particular logarithms[5]. Later in 1676 in his "epistola prior" to Leibniz he was further to improve the formal expression of the expansion, but this derivation by interpolation seems never to have been altered. The paradox remains that such Wallisian interpolation procedures, however plausible, are in no way a proof, and that a central tenet of Newton's mathematical method lacked any sort of rigorous justification (except in those few cases which could be checked by such existing techniques as algebraic division and root-extraction). Of course, the binomial theorem worked marvellously, and that was enough for the 17th century mathematician. The formal justification was the contribution of mathematicians from Euler to Abel using techniques not available to Newton, and was not over difficult. General praise has always been given to Newton for formulating the expansion. Shall we not continue to agree that it is fulsomely deserved though we can now see the inadequacies of his thought?

REFERENCES

1. See my article "Henry Briggs: The Binomial Theorem anticipated". *Math. Gazette*, Vol. XLV, pp. 9–12.

2. Compare (CUL. Add 3968.41:85) "In the beginning of the year 1665 I found the method of approximating series & the Rule for reducing any dignity of any Binomial into such a series . . . In those days I was in the prime of my age for invention & minded Mathematicks & Philosophy more than at any time since."

3. Especially by

J. M. Child: "Newton and the art of discovery," in "Isaac Newton, (1642–1727)." London, 1927:117–129;

and by

Jos. E. Hofmann: "Studien zur Vorgeschichte der Prioritätstreites zwischen Leibniz und Newton um die Entdeckung der höheren Analysis. 1. Materialen zur ersten Schaffensperiode Newtons (1665–1675)" = Abh. d. Preuss. Akad. d. Wiss. Math.-Naturw. Kl. 2. Berlin, 1943.

4. Add 3968.41:76.

5. Compare Add 4000:14v: ". . . (I) made these annotations out of Schooten & Wallis in winter between the years 1664 and 1665. At w^{ch} time I found the method of Infinite series. And in summer 1665, being forced from Cambridge by the Plague, I computed y^e area of y^e Hyperbola at Boothby in Lincolnshire to two & fifty figures." Such calculations, varying in length from 47D–57D, are to be found widely in Newton's 1665 manuscripts.

GROUP THEORY IN THE SIXTH FORM

By F. M. Hall

Much thought is being given at present to the possibility and desirability of introducing topics of modern mathematics, such as set theory and abstract algebra, into the sixth form curriculum. Very little of this kind of work has as yet been done in schools, chiefly because of its comparatively recent introduction into undergraduate courses, but it seems probable that it will come to play a part in the mathematical training of at least the abler boys in the fairly near future. Recently, during teaching practice, I had the opportunity of giving a course in group theory to four senior boys and I feel that the experience gained may be of interest to others who are contemplating similar kinds of work.

The four boys concerned were in their last year at school and had all gained awards at Oxford or Cambridge, two in mathematics and the other two in classics. None of them intended reading mathematics at the university, but they were all keen on learning something of advanced work in the subject while waiting to go up, and because of my fairly relaxed time-table as a student master I had the time necessary to prepare and give such a course. The two classics had done no mathematics above 'O' level, and I was a little apprehensive about dealing with this type of work with them, but I need not have worried. All four were extremely able and very keen to learn, and because of this and of the lack of prior technical knowledge required they made excellent progress. I took them in pairs, each pair seeing me for two hours a week throughout a ten week term.

The only book at all suitable for my purpose was that by P. H. Alexandroff, but for various reasons I preferred to teach from my own notes, which I prepared and distributed as I went along, managing to keep just ahead of the teaching.

I realised at the start that I must spend a lot of time on preliminaries. The concept of a group was completely strange to the pupils, and I was careful to lead up to it as gradually as I could. I introduced the idea of a set very early, and treated abstract algebra as a method of dealing with more general sets in the same way that elementary algebra deals with numbers (real or complex). The first step was therefore to discover exactly what ordinary algebra does. We found on analysis that it depends ultimately on the so-called 'Fundamental Laws of Algebra' dealing with the four rules or processes of addition, subtraction, multiplication and division, and I discussed the fundamental laws at length, going so far as to prove several simple results from the laws alone. These results (examples are the uniqueness of the zero, and the cancellation laws of addition

and multiplication) are obvious for ordinary numbers, but I made it quite clear that their importance lay in the fact of their being proved from the laws alone and therefore of their validity when we dealt with more general sets later. I think the boys appreciated this part of the course, since they were talking about ideas familiar to them but learning to think about them in a new way, and they were acquiring fairly painlessly some of the techniques of the new subject.

I pointed out in this work that the two really fundamental processes were those of addition and multiplication, and I took pains to show that in essentials these behaved alike. We easily came to the conclusion that here was the basis of ordinary algebra, and it would be so also in our extension to more general sets. By this time the boys were quite ready to appreciate the idea of a group as a set with a product satisfying certain laws.

I now gave the abstract definition of a group and explained it at some length. I used the multiplicative notation as is usual, reserving the additive for Abelian groups. (Alexandroff uses the additive throughout, in his book.) I talked about isomorphic groups straight away, and pointed out that as mathematicians our interest lay in the abstract structure of our groups, but that to fix our ideas and see some reason for studying them we would talk about plenty of concrete examples. This abstract outlook was possible with the boys I was dealing with, but might not be so with less intelligent pupils.

In proving elementary results such as the Cancellation Law I was covering familiar ground, and I quickly dealt with such matters as finite and infinite groups, Abelian groups, the inverse of a product and the powers and order of an element. Since this was probably the only course in modern mathematics that these boys would receive I frequently took the opportunity of talking about red herrings; for example the mention of infinite groups produced a memorable discussion on countable and non-countable infinities.

In order to drive home the ideas I had introduced, and to provide material for concrete illustrations of future concepts, I gave many examples and talked at length about them. These mostly belonged to a few basic types.

The first type included those whose elements are numbers of various kinds. A wide variety of groups can be formed in this way, and they have the advantage that there are no new concepts to be digested. Some examples which I used are as follows. The integers under addition, the real numbers under addition, or multiplication if we exclude zero, the rational numbers under addition all give infinite Abelian groups, as do the even integers under addition and the set of all numbers 2^n under multiplication. The last two were shown to be isomorphic to the group of integers. The numbers of the form $a + b\sqrt{n}$ give, for a and b rational (not both zero) and n a

fixed integer, a group under multiplication, but all such are not isomorphic. I also mentioned groups using complex numbers.

I gave several examples of finite cyclic groups, including the rotations of a regular polygon and the integers modulo n, under addition. The four numbers ± 1, $\pm i$ give a further example of the cyclic group of order 4. All the work on cyclic groups was quickly appreciated by the pupils. So far all our examples had been Abelian, but I now introduced the Dihedral Groups as the groups of transformations of regular polygons, and showed that the Dihedral Group of order 6 was non-Abelian. I laid particular emphasis on this, as the smallest non-Abelian group, and that of order 4 as an example of the Vierergruppe. Both are interesting, and have the advantage of possessing simple multiplication tables.

As more complicated examples I discussed the transformation groups of the regular polyhedra. The boys had some difficulty in appreciating the idea of an element being a rotation, but once understood these gave excellent material for illustrations throughout the course and were good examples for showing the complicated structure possible. I think I would always include these and the permutation groups in any course I gave on the subject, even though the boys in this instance did not always share my obvious delight in meeting groups with 'tons of structure'.

The permutation, or symmetric, groups gave the opportunity for demonstrating more isomorphisms, between the symmetric group of degree 3 and the dihedral group of order 6, and between the symmetric group of degree 4 and the group of transformations of the cube.

Finally I mentioned briefly the group of polynomials (chiefly for later use as an example of a ring) and also the direct sum of two or more groups. A good example of the latter is the Vierergruppe as the direct sum of two cyclic groups each of order two.

The description of the examples given took some time, but I considered it absolutely indispensable for any understanding of the course, and I felt that the boys found it interesting. Teachers will have their own favourite examples including some which occur in other branches of mathematics, and much experience is needed before we can select those which are best for our purpose. Before investigating further the structure of groups, I tried to indicate some of their uses. I found great difficulty in this, since the mathematical applications, for example in topology, could not be explained properly at this level, while the scientific ones would need somebody more familiar with them than I am. I would be interested in reading elementary accounts of such applications, if any are available.

Having discussed the group concept at such length I was able to start work on the general theory of their structure, and subgroups

formed an obvious beginning. The pupils easily appreciated this concept and the examples I gave, but I found much greater difficulty in the idea of a coset, which I had to introduce in order to prove Lagrange's Theorem, on the order of a subgroup of a finite group being a factor of the order of the whole group. This theorem was one of the highlights of the course, since it is fairly easy to prove, yet its importance can easily be seen and examples make it convincing. I used it to enumerate all groups of order up to 6 and to show that a group of prime order must be cyclic; this kind of work being concrete was appreciated.

With my next subject, that of invariant subgroups, I had much more trouble, due to the difficulty of giving easy examples and of showing the ideas underlying the work. I feel that this topic is so important that its explanation must be attempted in spite of its abstruseness.

I now discussed homomorphisms. This can be a fascinating study, but as I had not the time to develop the subject very far it was difficult to explain its importance. A very little work on projections was useful, and I was able to make a brief mention of exact sequences. Elementary homological algebra such as this, with perhaps a little diagram chasing, may enliven this type of course, but it is hard to avoid giving the impression of it being a mere juggling with symbols.

I thought that it would be useful at this stage, before leaving the subject of groups, to state some further results which it was not possible to prove properly. I hoped that this would show the boys some of the work that could be done if one went a little more deeply into the theory, and I think that it was successful in this. I mentioned the isomorphism theorems and the Jordan-Hölder Theorem, the latter being extremely difficult even to enunciate simply and not, therefore, very convincing. It did however provide a chance to mention soluble groups and the insolubility of the general quintic equation. Sylow's Theorems, on the other hand, were simple to state and were enthusiastically received.

I had intended to include a fair amount of work on rings and fields. Ever since I had given the idea of a group the pupils had been interested to know about other possible structures. I was planning the course very much as I went along, and I now discovered that the end of term was fast approaching, leaving me time for only a brief sketch of the definition of a ring and a field, and a few examples. As examples of rings I gave several sets of numbers, including that of even integers, the cyclic group of order n (a field if n is prime), polynomial rings and the ring of endomorphisms of an abelian group. Real numbers, complex numbers and quaternions gave further examples of fields. I was not satisfied with my cursory treatment,

and I would endeavour to give a much fuller account of these structures if I were to give the course again.

It will be seen that my difficulties always started when the subject matter became too abstract. Provided I could give plenty of concrete examples the boys were interested, but when this became difficult they could not appreciate the work nearly so much. Thus subgroups were quickly understood, but invariant subgroups and factor groups were not. On the whole they found the work fresh and stimulating, I believe, and could appreciate the abstract ideas provided they had an anchor with reality in the shape of illustrations. I was interested to notice a difference in attitude between the mathematicians and classics. The former had the advantage of being more familiar with the mathematical outlook, but the classics brought to bear a freshness of intellect and a breadth of view which caused them sometimes to experience a keener pleasure and greater sense of discovery than the others. These were, I must emphasize, exceedingly intelligent boys, and I would not advocate making a course in group theory compulsory for all classical sixth forms!

As to the content of the course, I feel that I was right to spend a long time on the introduction and examples of groups, and I would have liked to have dealt with rings and fields in similar detail. Of further topics subgroups form an obvious choice, and homomorphisms should at least be mentioned. There is probably also scope in some subjects that I did not touch, for example factorization theory.

I do not wish to discuss here to what extent this type of work should replace some of the present sixth form curriculum. It seems likely that a shift of emphasis will come, from the technically complicated but often trivially important studies which we have at present towards a course which will give the pupils a greater insight into modern mathematics and its methods and a better training for their minds. But it will need a lot of thought and experiment before we know how best to treat such branches of the subject in our schools.

Dulwich College F. M. H.

MECHANICS IN SCHOOL AND UNIVERSITY

By R. Buckley

Mechanics is a branch of applied mathematics taught in both school and university. From the mathematical standpoint, the subject is learned in schools primarily by the method of solving examples. Theoretically, this procedure is supposed to help the pupil to grasp the fundamental principles and obtain for him a good examination result. That these seem to be nothing like as successful as they could be is one reason which prompts me to write this article. Others, including the pupil's lack of appreciation of the subject, will be apparent in the points I shall make. These points have arisen from some years' experience of lecturing to first year undergraduates and from marking advanced level scripts in applied mathematics. The efforts met with in both instances have led me to believe that there is a case for comparing the knowledge and understanding of the average sixth form pupil with that normally required of a freshman about to read applied mathematics for a general degree. On the one side is the ex-pupil, drilled in the method of solving examples and often lacking the understanding of the background principles involved; on the other is the undergraduate needing to draw upon an understanding and a background wider than he usually possesses. I feel sure that the disparity so defined exists not so much because of a lack of bulk material taught in the schools, but rather because the presentation is misguided and the emphasis misplaced.

My aim is to present and discuss certain topics which, I believe, should be an integral part of the school teaching system in mechanics if the pupil is to gain the understanding and grasp of the subject which are so necessary later on. The suggestions made would perhaps curtail the 'drilling time' but would offer instead, understanding. In mechanics, where analysis of the question is so important, this is the only reliable basis for problem solving. We are led to the first item:

Analysis of the question

This should not be thought of as simply 'reading the question,' which may suffice in many cases, e.g. solving equations, evaluating determinants, integrals and so on, but here the situation is different. The pupil must realise that, in most cases, the problem is a written description of some physical system and, from the reading, he should be able to draw it, represent it or get a mental picture of it. He should further realise that the problem is determinate; the information necessary to the solution is there if the problem is interpreted

correctly. By **analysis of the question,** which goes naturally with the interpretation, I mean the ability to sort out what is given and to decide what principles are to be used in the solution. From time to time I have found it illuminating to take a problem with a class and go through the steps mentioned. It is surprising to discover the difficulties the students have in interpreting the various pieces of information given. Many examples come to mind, e.g. the inequalities and boundary conditions given to restrict the problem; the acceleration being variable and so preventing the use of such laws as $v = u + ft$; the breaking of contact between two systems such as a beam just tilting off one support or a particle leaving a surface, implying a reaction is zero, or conversely. These and many others are being constantly overlooked or misused in solutions offered.

Degrees of freedom

A fundamental part of the analysis of many applied mathematics problems is deciding the number of degrees of freedom of the physical system or the number of unknown quantities. The pupil who realises the importance of these and states them is extremely rare, whilst one is continually meeting those who write down every equation they can think of; they have been taught to 'resolve horizontally, vertically and take moments' and this they intend to do! Yet the teaching of degrees of freedom and constraints lends itself so readily to physical illustration, whilst offering so much help in deciding the number and choice of variables and in expressing the analysis in mathematical form, that I would have thought it to be a natural topic for a sixth form. Instead, I find that most freshmen are unaware of the ideas.

The wider picture

Mechanics, like all applied mathematics, draws upon many techniques in its theory and practice. If the subject is to be fully appreciated, then it must be set against a wide backcloth. To me it seems wrong to give the pupil a set method of solving a particular type of example without at least suggesting other methods. For example, when statics was a subject for the general degree, I found it almost impossible to persuade some students that other and more appropriate methods often existed than resolving and taking moments. Geometry and analysis were not a part of their backcloth; they were not mechanics. But configuration and relative positions of bodies are so automatically a part of all applied mathematics that it seems natural to call upon geometry as an aid to working. In this context, one realises the value of Lamb's Statics as a teacher's text; here the fundamental postulates are so clearly stated, and the

various branches of mathematics are called upon to dovetail into the general plan. This is what I mean by appreciation of the subject—it gives the pupil a wider angle of vision and a flexibility so important later on. Indeed, it could be safely argued that, as statics vanishes from the university syllabuses, its importance in schools lies in giving to the pupil this flexibility of approach. Let us examine a few of the methods taught.

(a) *Vectors*

We are all aware of the demand for a knowledge of vectors and it is evident that it is now being taught in many schools. But let us be careful of what we teach and how we teach it. The majority of pupils using it in examination show that they do not realise the difference between free and line vectors. One suspects that they are taught vectors, and it matters not whether they are dealing with a particle or a plane system. The theory of line vectors is not simple and should not, I think, be taught in schools; but the pupil must be made aware of the distinction, and this is not difficult. It is, for example, only necessary to move a force parallel to itself and thereby show the different effects it has on a rigid body.

(b) *Virtual Work*

Whether it is wise to teach this to a sixth form—scholarship candidates excepted—I cannot be sure. It is certainly one of the most difficult parts to teach. Perhaps it is possible to apply it to a particle, but the extension to plane systems requires more previous knowledge than those candidates exhibit whose attempts I have marked. I am sure this is why the majority of such efforts are ill-disciplined and often pathetic. Before embarking on the extension mentioned above it is *essential* that the student should know about degrees of freedom and some plane kinematics, and should appreciate what is meant by a virtual displacement. Certainly it is a case of a little knowledge being a dangerous thing. To illustrate this, here are one or two examples I have met: a system is given a deformation before it is made deformable; no knowledge of how to choose the displacement leading to the best solution; no realisation of the number of degrees of freedom brought into play by the displacement given or of the restriction sometimes placed on it and usually represented by a differential form.

It would be interesting to read other views on the methods and success of teaching this most elegant principle which is so often required in the more advanced work in many branches of applied mathematics.

(c) *Plane Kinematics*

It is puzzling to find that a large number of students have learnt some rigid dynamics at school without doing any plane kinematics. The ideas of translation and rotation, their composition to form a

general displacement, and their relevance to degrees of freedom form an important topic. Moreover, it is one which lends itself so readily to practical illustration that I would make a strong plea for the teaching of something along the lines mentioned, to give a background to the later work on rigid dynamics.

(d) *Friction*

School problems on statical friction may be divided, for our purpose into two types; those on limiting equilibrium and those on equilibrium. Many pupils misread or deliberately misinterpret the latter for the obvious reason that they do not like dealing with inequalities. And yet a geometrical interpretation will give them, in most cases, a simple and clear-cut method of solving the problem. This seems a good opportunity, so often lost, of drawing other techniques and branches of mathematics into the general picture.

The selection of points above, and readers will be able to add others, leads me to this conclusion. Mechanics will probably hold a place in school and university syllabuses for a long time to come, if only as the most convenient medium for the teaching of many important techniques and for bringing together different branches of pure and applied mathematics. Geometry, algebra, analysis, vectors and so on can and must all be made to play their part for the good of the whole, or the subject loses most of its present-day purpose.

Whilst being more interested myself in understanding and appreciating the subject I have tried to keep in mind throughout the desire, naturally held by a teacher, to produce good problem solvers. I do not believe, however, that the latter can exclude the former; instead, only together are they of any real value to the teacher, pupil, graduate and future teacher. In 'The Teaching of Mathematics' occurs the statement—"the gradual evolution of the professional mathematician and the arrangement of university work to keep pace with the ever increasing advances in mathematics have consequences which can hardly be assessed, beyond the bare statement that future teachers in secondary schools are unlikely to be drawn from those who have attained high places in their university work." If this is true, then there must be a change of heart in others besides those who set examination papers.

University of Leicester R. B.

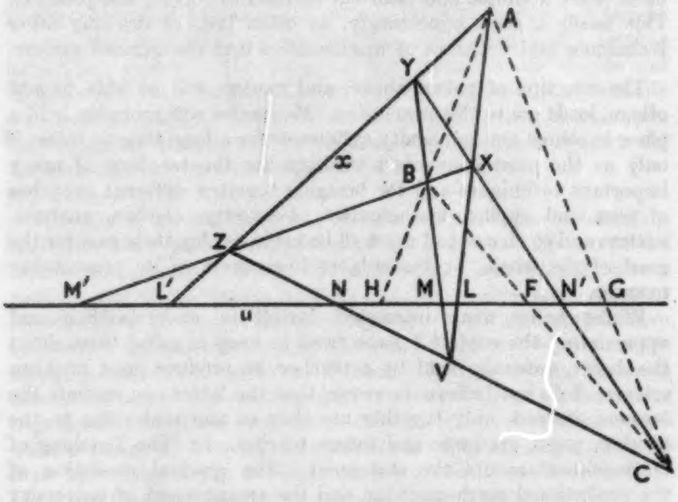
A PROBLEM IN PROJECTIVE GEOMETRY

By J. F. Rigby

In his article "A Problem in Projective Geometry" (*Gazette*, December 1960) T. G. Room says that he has not been able to find a projective geometric proof of his theorems. May I give the following proof?

Using the notation of the original article, Theorem I may be re-stated thus:

Let L, L'; M, M'; N, N' be three distinct pairs of points* of an



involution $\mathbf{J}$ on a line u. Let P be an arbitrary point of u. Then, if $Q = (L, L')/P$; $R = (M, M')/Q$; $S = (N, N')/R$, the projectivity $\mathbf{Z}$ determined on u by the mapping $P \to S$ is an involution. $\mathbf{Z}$ has two double points K and K' which are rationally constructible from the three original pairs of points of $\mathbf{J}$. K and K' are also a pair of points of $\mathbf{J}$.

We must not assume that, in the plane under consideration, every projectivity on a line has at least one self-corresponding point, nor the equivalent result that every line meets a given conic. We shall prove the theorem in two parts.

* This is the author's original phrase, but the most important requirement is that the two points of each pair shall be distinct. If two or more pairs coincide we simply obtain a special case of the theorem.

Lemma 1. The projectivity $\mathbf{Z}$ has two distinct self-corresponding points K and K' which are rationally constructible, and K, K' are a pair of points of $\mathbf{J}$.

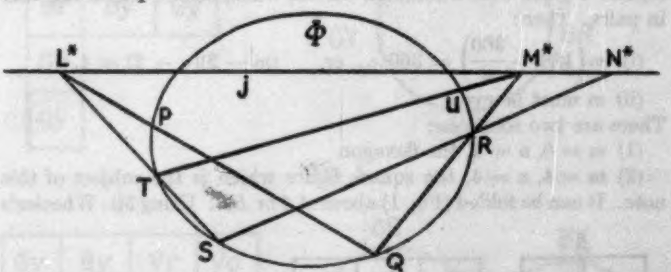
Proof. Let V be any point not on u and let x be any line through L' distinct from u. Let $x \cap VM = Y$, $x \cap VN = Z$. Let $M'Z \cap VL = X$. Then by the quadrangle theorem for involutions XY passes through N'. Let $VX \cap YZ = A$ and define B, C similarly. Then ABC is the harmonic triangle of the quadrangle $VXYZ$. Let BC, CA, AB meet u in F, G, H respectively.

Now $A(XY, CB)$ is a harmonic pencil, by the harmonic property of a quadrangle. Hence (LL', GH) is a harmonic range on u. Hence $H = (L, L')/G$. Similarly $F = (M, M')/H$ and $G = (N, N')/F$. Hence $\mathbf{Z}$ maps G into itself.

Let F', G', H' be the mates of F, G, H in the involution $\mathbf{J}$. Then (LL', GH) is a harmonic range, so $\mathbf{J}(LL', GH) = (L'L, G'H')$ is a harmonic range. Hence $H' = (L, L')/G'$. Similarly $F' = (M, M')/H'$ and $G' = (N, N')/F'$. Hence $\mathbf{Z}$ maps G' into itself. G and G' are distinct, for if G were a double point of $\mathbf{J}$ then $\mathbf{J}$ would have a second double point G^* distinct from G. We can easily show that $\mathbf{Z}$ would map G into G^*, a contradiction. K and K' are the points G and G'.

Lemma 2. $\mathbf{Z}$ is an involution.

Proof. It is sufficient to prove the result for points on a non-singular conic Φ instead of on a line u. We shall use the same letters to denote the points.



Since L, L'; M, M'; N, N' are pairs of an involution $\mathbf{J}$ on Φ, LL', MM', NN' are concurrent in the point J say. Let j be the polar of J with respect to Φ. Then the poles L^*, M^*, N^* of LL', MM', NN' with respect to Φ all lie on j. Now a pair of points D, D' on Φ harmonically separates the pair L, L' if and only if DD' passes through L^*. Similar results hold for the pairs M, M' and N, N'. Hence if P lies on Φ and if L^*P meets Φ again in Q, M^*Q meets Φ again in R and N^*R meets Φ again in S, then the projectivity $\mathbf{Z}$ maps

P into S. Let L^*S meet Φ again in T and let M^*T meet Φ again in U. Let UP meet RS in E. Applying Pascal's Theorem to the hexagon $PQRSTU$ we see that L^*, M^*, E are collinear, whence we deduce that $E = N^*$. Hence N^*U meets Φ again in P. Hence $\mathbf{Z}$ maps S into P. Hence $\mathbf{Z}$ interchanges the pair of points P and S, and so $\mathbf{Z}$ is an involution unless $\mathbf{Z}$ is the identical mapping.

If $\mathbf{Z}$ is the identical mapping, then $P = S$ for all positions of P. Consider the special case when $Q = M$. Then $R = M$ also, and (LL', PM) and (NN', PM) are both harmonic ranges. Hence P and M are the double points of $\mathbf{J}$, so $M = M'$ which contradicts the conditions of the theorem. Hence $\mathbf{Z}$ is not the identical mapping and so is an involution.

University College, Cardiff J. F. R.

Editorial note. A solution was also received from E. A. Maxwell.

SQUARE FLEXAGONS

By P. B. Chapman

The flexagon was defined by Mr. R. F. Wheeler (M. G. 1958, p.1) as a strip of equilateral triangles which can be folded in various ways into a hexagon to exhibit different colour combinations. Can this idea be generalised? Suppose a plane figure to consist of m regular n-gons, with a common vertex, which can be folded together in pairs. Then:

(i) $m\left(180 - \dfrac{360}{n}\right) = 360$ or $(m-2)(n-2) = 4$,

(ii) m must be even

There are two solutions:

(1) $m = 6$, $n = 3$, the flexagon

(2) $m = 4$, $n = 4$, the square figure which is the subject of this note. It can be folded (Fig. 1) about AA or BB. Using Mr. Wheeler's

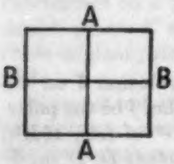


Fig. 1

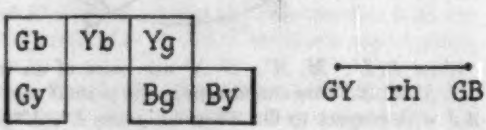


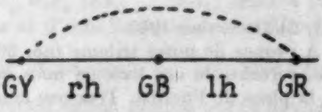
Fig. 2

terminology, strips and maps are drawn: to assemble a strip, fold together adjacent faces of like colour (small letters denote colours

on the back) until two colours are left which correspond to a point on the map, then join together the thick lines.

The principle of designing a strip is the insertion of a new colour at opposite corners of an existing square or strip, shown in Fig. 2. R. h. denotes the "sense" of the folds: the strip for the l.h. version is the mirror image of that shown. GY and GB are "corner points" at which further insertions are possible. If all of them are in the same sense the resulting map is a simple chain of links, but whenever two consecutive links have opposite senses cyclic effects occur:

(a) *if the links have the same common colour* transition can be made from GY to GR by turning over two loose flaps:—



(b) *if the links have different common colours* two more links appear, forming a proper cycle, shown in Fig. 3. (Cf. the flexagon, where

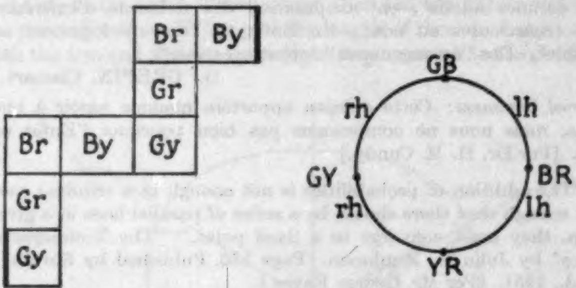


FIG. 3

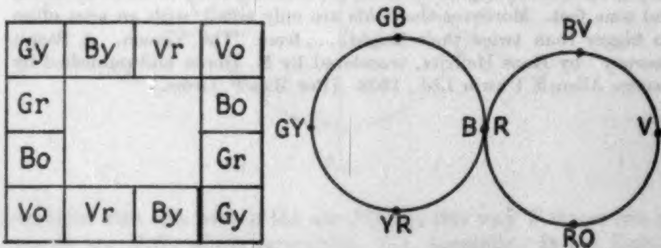


FIG. 4

every insertion produces a cycle and every alternation of sense produces flaps). In this cycle, only GY and BR are corner points. By continuing $BR—BV$ (l.h. to avoid flaps), $BV—VO$ (r.h.) etc. a chain of cycles can be formed. If their number is even, as in Fig. 4, the strip is the border of a square which is not altered topologically in assembly, so that cutting and joining is not strictly necessary.

King Edward's School, Birmingham P. B. C.

GLEANINGS FAR AND NEAR

1961 Paris MATCH, 24 December 1960

Editorial Note: À propos de notre tribune (no. 606) sur la pluralité des mondes habités, certains de nos lecteurs nous demandaient quelle est dans tout cela, la place de Paradis. D'autres leur répondent.

Letter: Au Jugement dernier la Terre étant une boule, le volume du ciel qui l'environne sera fonction de πR et, par conséquent, infini. Tous les élus pourrent y prendre place. Le cube "in inferno" étant, au contraire, immuable fatalement, il ne peut y avoir en Enfer "que trés peu de damnés admis", en comparaison des milliards d'existences réalisées (conscientes ou non). En Enfer, la crise du logement est "inévitable". Des "arrangements" seront nécessaires.

G. CREPIN, Clamart

Editorial Comment: Cette opinion apportera quelque espoir à bien des gens, mais nous ne comprenons pas bien pourquoi l'Enfer est cubique. [Per Dr. H. M. Cundy.]

1962. "The addition of probabilities is not enough in a criminal case: it is not enough that there should be a series of parallel lines in a given direction, they must converge to a fixed point." 'The Technique of Advocacy' by John H. Munkman. Page 136. Published by Steven & Sons Ltd., 1951. [Per Mr. George Eaves.]

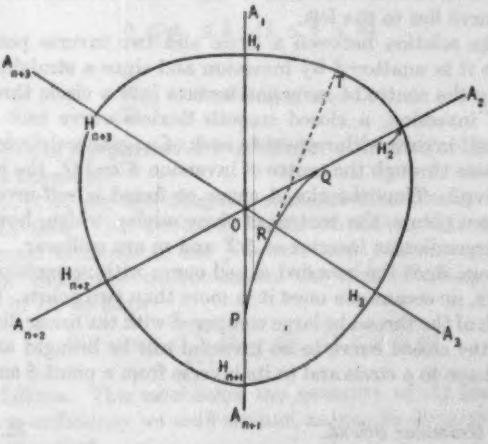
1963. On the lower levels the terraces are about three feet high, but they grow steadily higher towards the summit, most of them up to six and nine feet. Moreover the fields are only small, with an area often no bigger than twice their height, . . . from "The Yemen. A Secret Journey" by Hans Helfritz, translated by M. Heron and published by George Allen & Unwin Ltd., 1958. [Per Miss F. Gross.]

A METHOD FOR CONSTRUCTING A ROUNDED CLOSED FLEXLESS CURVE WHICH IS SELF-INVERSE WITH RESPECT TO n *POINTS AND YET IS NOT A CIRCLE*

By H. G. Woyda

In Note **2949** printed in the May 1961 issue of the *Gazette*, I gave a method for constructing a rounded closed flexless curve, not necessarily a circle, which is self-inverse with respect to two points. The following is a method for constructing such a curve which is self-inverse with respect to n points:

Let $2n$ rays $OA_1, OA_2, OA_3, \ldots OA_{2n}$ from a point O in a plane divide the angle 2π at O into $2n$ equal parts. These form n axes at O spaced successively at an angle $\frac{\pi}{n}$. The following is a method of constructing a rounded closed flexless curve having these n axes as axes of symmetry and of finite (non-vanishing) curvature at each point: Let PQ be any smooth curve without a flex (but not necessarily of finite radius of curvature at every point) touching OA_{n+1} at P and OA_2 at Q. Let a thread longer than the arc PQ be attached to the point P and let PH_1 equal the length of thread. Let the thread with the free end T initially at H_1 be kept taut while the portion PR



coincides with this part of the arc PQ. In this way T traces out a smooth arc H_1H_2 which meets OA_1, OA_2 normally. It has finite, non-vanishing curvature at every point, since TR is the radius of

curvature at T. It has no flex. By reflecting this arc successively with respect to the various axes, a smooth flexless closed curve $H_1H_2H_3 \ldots H_{2n}$ is derived with n axes of symmetry A_1OA_{n+1}, $A_2OA_{n+2}, \ldots \ldots, A_nOA_{2n}$.

As a point and its image in a straight line can be regarded as inverse points with respect to the straight line regarded as a circle of infinite radius, the rounded closed, flexless curve generated as above (though not a circle) can be regarded as self-inverse with respect to each of n straight lines.

Let now the closed curve be inverted from a point S which is not on an axis and is outside $H_1H_2H_3 \ldots H_n$ and also outside each of its circles of curvature. (The circle of curvature at every point is known thanks to the above method of generating the arc H_1H_2). The relation between a curve and its circle of curvature at any point (with three "consecutive" points common to both) is clearly unaltered by inversion. Hence if the centre of inversion be inside a circle of curvature at one point but outside that at another point of a smooth flexless curve, then the inverse curve must have at least one point of inflexion between the inverses of the two given points. Also, as one traverses the closed curve $H_1H_2H_3 \ldots H_{2n}$ clockwise, the circle of curvature at any point X of the curve is always on the right of the tangent to it at X. Hence, as one traverses the inverse curve with respect to the point S, (a point outside every circle of curvature) in an anti-clockwise direction, every circle of curvature of the inverse curve lies to the left.

Now the relation between a circle and two inverse points with respect to it is unaltered by inversion and since a straight line not containing the centre of inversion inverts into a circle through the centre of inversion, a closed smooth flexless curve (not a circle) which is self-inverse with respect to each of n (genuine) circles each of which passes through the centre of inversion S and O', the inverse of O, is derived. Thus the closed curve so found is self-inverse with respect to n points, the centres of these circles, which, however, lie on the perpendicular bisector of SO' and so are collinear.

Moreover, since the rounded closed curve with no multiple points has no flex, no secant can meet it in more than two points. Clearly if the length of the thread be large compared with the linear dimensions of OPQ, the closed curve to be inverted can be brought arbitrarily close in shape to a circle and to its inverse from a point S sufficiently remote.

Kingston Grammar School. H. G. W.

DIAGONAL ELEMENTS OF DOUBLY-STOCHASTIC MATRICES

By I. P. Gorton

A square matrix is said to be *doubly-stochastic* if its elements are non-negative and if all row-sums and column-sums are equal to 1. The study of doubly-stochastic matrices was initiated by I. Schur [**4**] and was subsequently taken up by Hardy, Littlewood, and Pólya, who proved the following fundamental proposition [**1**, Theorem 46]. *If* $(x_1, \ldots, x_n)$ *and* $(y_1, \ldots, y_n)$ *are any given real vectors, then there exists a doubly-stochastic matrix* D *such that*

$$(y_1, \ldots, y_n) = (x_1, \ldots, x_n)D$$

if and only if

$$\bar{y}_1 + \ldots + \bar{y}_k \begin{Bmatrix} \leq \\ = \end{Bmatrix} \bar{x}_1 + \ldots + \bar{x}_k \begin{Bmatrix} 1 \leq k < n \\ k = n \end{Bmatrix}.$$

Here $\bar{x}_1, \ldots, \bar{x}_n$ denote the numbers $x_1, \ldots, x_n$ arranged in non-ascending order of magnitude and $\bar{y}_1, \ldots, \bar{y}_n$ are defined analogously. Making use of the theorem of Hardy, Littlewood, and Pólya, A. Horn [**2**, Theorem 9] gave an ingenious proof of the following result.

THEOREM. *The numbers* $a_1, \ldots, a_n$ *are the diagonal elements of a doubly-stochastic* $n \times n$ *matrix if and only if they satisfy the conditions*

$$0 \leq a_k \leq 1 \quad (1 \leq k \leq n) \tag{1}$$

and

$$\sum_{k=1}^{n} a_k - 2 \min_{1 \leq j \leq n} a_j \leq n - 2. \tag{2}$$

My object in the present note is to offer a simple and direct proof of Horn's theorem.

If $a_1, \ldots, a_n$ are the diagonal elements of a doubly-stochastic matrix D, then (1) is obviously valid. Moreover, the sum of the non-diagonal elements in the j-th row and j-th column of D is $2(1 - a_j)$. Since this cannot exceed the sum of *all* non-diagonal elements in D, we have

$$2(1 - a_j) \leq \sum_{k=1}^{n} (1 - a_k) \quad (1 \leq j \leq n)$$

and (2) follows. This establishes the necessity of (1) and (2). To prove their sufficiency we shall assume, as may be done without loss of generality, that

$$a_1 \geq a_2 \geq \ldots \geq a_n. \tag{3}$$

It is then possible to choose non-negative numbers $b_1, \ldots, b_{n-1}$ such

that in the matrix $S = (s_{ij})$, defined by the equations

$$s_{ii} = a_i \quad (1 \le i \le n), \quad s_{ij} = b_{\min(i,j)} \quad (i \ne j),$$

the first $n - 1$ row-sums and the first $n - 1$ column-sums are equal to 1. For suppose that $b_1, \ldots, b_k$ have already been chosen; then

$$b_1 + \ldots + b_k + a_{k+1} \le b_1 + \ldots + b_k + a_k \le 1$$

and therefore b_{k+1}, too, may be chosen so as to meet our requirements.

We note that

$$\sum\begin{Bmatrix}1 \le i, j < n \\ i \ne j\end{Bmatrix} s_{ij} = n - 2 - \sum_{k=1}^{n} a_k + a_{n-1} + a_n$$

and so, by (2) and (3),

$$\sum\begin{Bmatrix}1 \le i, j < n \\ i \ne j\end{Bmatrix} s_{ij} \ge a_{n-1} - a_n. \tag{4}$$

Next, we choose a matrix $T = (t_{ij})$ such that $t_{ii} = 0$ $(1 \le i \le n)$ and

$$\sum\begin{Bmatrix}1 \le i, j < n \\ i \ne j\end{Bmatrix} t_{ij} = a_{n-1} - a_n, \tag{5}$$

$$0 \le t_{ij} \le s_{ij} \quad (1 \le i, j < n;\ i \ne j), \tag{6}$$

$$t_{in} = -\sum_{j=1}^{n-1} t_{ij}, \quad t_{ni} = -\sum_{j=1}^{n-1} t_{ji} \quad (1 \le i < n).$$

The choice of t's satisfying (5) and (6) is possible in view of (4).

It is now a matter of straightforward verification that the matrix $S - T$ is doubly-stochastic and has $a_1, \ldots, a_n$ as its diagonal elements. This completes the proof.

It may be added, in conclusion, that it is possible to establish an analogue of Horn's theorem for infinite doubly-stochastic matrices and, indeed, for a still wider class of infinite matrices. For details we refer the reader to [3].

The University, Sheffield. I. P. G.

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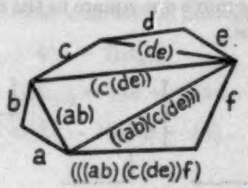
SOME PROBLEMS IN COMBINATORICS

By H. G. Forder

The following problems are merely different forms of one:

1. In how many ways can a convex polygon of $n + 1$ sides be split into triangles by non-intersecting diagonals? We represent such a splitting by a symbol. Take one side of the polygon as base and letter the other n sides in order by $a, b, c, \ldots.$ If a triangle with sides lettered x, y is cut off, denote the diagonal by (xy). If this diagonal and side z form the sides of another triangle of the dissection, denote the third side of that triangle by $((xy)z)$. Similarly for two diagonals, as the figure illustrates. Finally the symbol for the unlettered base is the symbol for the dissection.



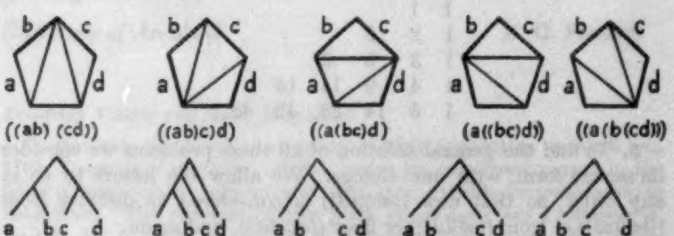
Such a symbol then has the following form. We have n letters in a *fixed* order; any two adjacent letters can be combined, then any two adjacent combinations, or a combination and an adjacent letter, to left or right, can be combined together. Thus our problem is:

2. How many "binary associations" can be formed from n letters in fixed positions?

These binary associations can be represented by trees.

3. How many trees can be formed with n ends, each fork having two branches?

We illustrate these transformations by a simple case.



The symbol in 2 can be simplified so that the old and the new are in one-to-one correspondence, as follows. We may omit the right

hand bracket, writing

$$((ab(cd \quad (((abcd \quad ((a(bcd \quad (a((bcd \quad (a(b(cd$$

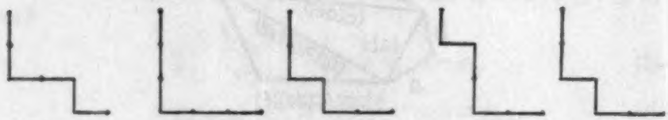
This reminds one of the Polish notation in symbolic logic. Furthermore we can omit the last letter, getting

$$((ab(c \quad (((abc \quad ((a(bc \quad (a((bc \quad (a(b(c$$

This abridged form can be completed in just one way to make the earlier form.

It will be seen that now the number of brackets equals the number of letters and that at any stage in traversing the symbol from left to right the number of brackets is never less than the number of letters.

Now take a chess board. Start from the top left square and interpret the symbol as follows: a bracket means, move down one square; a letter means move one square to the right. Thus our moves are in the above cases:



((ab(c (((abc ((a(bc (a((bc (a(b(c

Our problem has now been transformed to:

4. How many paths of this kind starting at the top-left corner end on the diagonal?

If we note that each square can be reached either from the square just above it or from the square on its left, we can construct the following table for the number of paths to any square below the diagonal.

1					
1	1				
1	2	2			
1	3	5	5		
1	4	9	14	14	
1	5	14	28	42	42.

5. To find the general solution of all these problems we consider its second form, with one change. We allow the letters to be in any order, so that now $((ab)(cd))$ is considered as distinct from $((ba)(cd))$ or from $((ab)(dc))$ or from $((ca)(bd))$, and so on.

Suppose now we can make A_n associations from n letters. How are A_n, A_{n-1} connected? If an association of $n - 1$ letters is given, a new letter x can be inserted at the beginning, or at the end, or internally.

In the first two cases we get $(x(\ldots))$ and $((\ldots)x)$, the part $(\ldots)$ denoting any association of $n-1$ letters. These give then a contribution $2A_{n-1}$ to A_n.

To consider internal insertions, suppose that at some stage in the building up of the association of $n-1$ letters we have constructed $((P)(Q))$ where P, Q are letters or associations of some of the $n-1$ letters. Thus P might be $(((pq)(rs))t)$, Q might be $(u(vw))$.

Then x can be inserted in four ways

$$(((x(P))Q), \quad ((((P)x)Q), \quad ((P)(x(Q)), \quad ((P)((Q)x)).$$

But if we have $n-1$ letters the association is formed by making $n-2$ successive binary associations, and thus we get $4(n-2)$ associations, when x is inserted, for each one of the old, i.e. in all $4(n-2)A_{n-2}$ associations.

Hence, finally, the total number of associations of n letters is

$$A_n = 2A_{n-1} + 4(n-2)A_{n-1}$$

$$A_n = (4n-6)A_{n-1} = (4n-6)(4n-10)A_{n-2}$$

$$= (4n-6)(4n-10)(4n-14)\ldots(4r-2)A_r$$

But $A_1 = 1$. Hence

$$A_n = 2 \cdot 6 \cdot 10 \ldots (4n-6) = 2^{n-1} \cdot 1 \cdot 3 \cdot 5 \ldots (2n-3)$$

Now in problem 2 the order of the letters was fixed. Hence to obtain the number B_n of solutions for that problem we divide A_n by $n!$

$$B_n = \frac{2^{n-1}(2n-2)!}{n!\,2 \cdot 4 \cdot 6 \ldots (2n-2)} = \frac{(2n-2)!}{n!(n-1)!}.$$

This then is the number of ways of dissecting an $\overline{n+1}$—gon as required in problem 1, and all the problems are solved e.g. $n = 6$ gives $B_n = 42$.

University of Auckland. H. G. FORDER

Friedrich Vieweg and Sons, Brunswick

To celebrate the 175th anniversary of their foundation on April 1st., 1786, Friedrich Vieweg are offering prizes for important papers in the fields of mathematics, physics and chemistry, suitable for publication in book form. The curators of this prize endowment are Professors J. Bartels, W. Gerlach, W. Haack, R. Huisgen, J. Mattauch, W. Quade, F. Sauter, F. Seel, H. Siedentopf and E. Wicke.

The publishers will be glad to send on request details of the competition to interested readers.

ON THE D CALCULUS FOR LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

By A. Robinson

1. *Introduction.* While the operational calculus for the solution of *initial value problems* for linear differential equations with constant coefficients is now commonly introduced in a thoroughly satisfactory manner—either by means of the Laplace transform (e.g. ref. 1) or by the method of J. Mikusinski (ref. 2)—the same cannot be said of the simpler method for the *general integration* of such equations which is known as the D calculus. In fact, I cannot recall any text which gives a consistent definition of the meaning of expressions of the type $[F(D)/G(D)]y$, where F and G are polynomials. The reason for this state of affairs seems to be that the scope of the D calculus is rather limited so that the validity of the result can be verified in each case (e.g. ref. 3). In particular, this applies to the decomposition of an operator $F^{-1}(D)$ into partial fractions, which is the central step in the solution of an equation with nonvanishing right hand side of general form. Nevertheless a rational approach to the entire problem is perhaps not out of place. This is attempted in the present paper.

2. *Basic Definitions.* Let C^0 be the space of complex continuous functions of a real variable x defined in an interval I, which may be open, closed, etc. and C^n the space of complex functions of x with continuous nth derivatives. The elements of C^0 will be denoted by y, z, v, w, with or without indices. Let J be the field of complex numbers. Elements of the ring $J[D]$ i.e. polynomials of the "indeterminate" D with complex coefficients will be denoted by F, G, H, K, with or without suffices, and elements of the field $J(D)$ i.e. rational functions of D with complex coefficients will be denoted similarly by f, g, h, k, Let $F(D) \in [J(D)]$ be a polynomial of degree not exceeding n, and let $y \in C^n$. Then we denote by Fy the element of C^0 which is given by $F(d/dx)y$.

An ordered pair of element $y, z \in C^0$ will be denoted by $y{:}z$. We say that $y{:}z$ *belongs to* f, $f = G/H$, $G, H \in J[D]$ if there exists a $w \in C^n$ such that $y = Gw$, $z = Hw$. In order to show that this definition is legitimate we have to establish that it is independent of the particular representation of f as a fraction of elements of $J[D]$. Let $f = G/H = G'/H'$, then $f = G \,.\, G'/H \,.\, G' = G' \,.\, G/H' \,.\, G$ and so it is sufficient to show that the same pairs belong to f independently of whether we use in the definition the function G/H or another function GK/HK, $K \in J[D]$. Now if $y = Gw$, $z = Hw$, then there

exists an element $w' \in C^\infty$ such that $w = Kw'$. Hence $y = GKw'$, $z = HKw'$, so that $y{:}z$ belongs to f also according to the representation $f = GK/HK$. Again, if it is known that $y{:}z$ belong to $f = FK/GK$ then there exists a w such that $y = FKw$, $z = GKw$. Hence, putting $w' = Kw$, we have $y = Fw'$, $z = Gw'$, $y{:}z$ belongs to f also according to the representation F/G. This shows that the definition is legitimate. We write $y{:}z < f$. Note that if f reduces to a polynomial, $f = F/1 = F$, then $y{:}z$ belongs to f if and only if $y = Fz$.

For the sequel, we require the following

2.1. *Lemma*. Let F, $G \in J[D]$, $G \neq 0$, and suppose that F and G have no roots in common. Assume that for some $y \in C^\infty$

2.2. $$Fy = 0$$

Then there exists a function $z \in C^\infty$ which satisfies simultaneously

2.3. $$Gz = y, \quad Fz = 0$$

Proof. If $F = 0$ then the result is trivial. Suppose that $F \neq 0$. Let the different roots of F be $w_1, \ldots, w_k$, with multiplicities $n_1, \ldots, n_k$ respectively. Then y is of the form

$$y = \sum_{i=1}^{k} P_i(x)e^{w_i x}$$

where $P_i(x)$ is a polynomial of degree $n_i - 1$ at most, $i = 1, \ldots, k$. We have to determine z in the form

$$z = \sum_{i=1}^{k} Q_i(x)e^{w_i x}$$

where the $Q_i(x)$ also are of degree $n_i - 1$ at most, such that

2.4. $$G(Q_i(x)e^{w_i x}) = P_i(x)e^{w_i x}, \quad i = 1, \ldots, k$$

Now let $G(D + w_i) = H(D)$, then according to the "rule of exponential shift", which can be verified easily,

$$G(Q_i(x)e^{w_i x}) = e^{w_i x}HQ_i(x)$$

It follows that 2.4 will be satisfied provided

2.5 $$HQ_i(x) = P_i(x), \quad i = 1, \ldots, k.$$

Let $H(D) = a_0 + a_1D + \ldots + a_mD^m$ where $a_0 = H(0) = G(w_i) \neq 0$, by assumption. Then 2.5 becomes for any particular i,

2.6 $$\left(a_0 + a_1\frac{d}{dx} + a_2\frac{d^2}{dx^2} + \ldots + a_m\frac{d^m}{dx^m}\right)$$
$$(b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \ldots + b_n)$$
$$= c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \ldots + c_n$$

where

$$P_i(x) = c_0x^n + c_1x^{n-1} + \dots + c_n$$

and

$$Q_i(x) = b_0x^n + b_1x^{n-1} + \dots + b_n, \quad n = n_i - 1$$

so that 2.6 represents a condition for the coefficients of $Q_i(n)$. Expanding the left hand side of 2.6 and comparing coefficients we we obtain the equations

$$\begin{aligned} a_0b_0 &= c_0 \\ a_0b_1 + na_1b_0 &= c_1 \\ a_0b_2 + (n-1)a_1b_1 + n(n-1)a_2b_0 &= c_2 \end{aligned}$$

from which $b_0, b_1, \dots, b_n$ can be determined successively. This completes the proof of the lemma.

3. *Fundamental properties*

The following implications are of importance in the application of the D calculus to the solution of differential equations with constant coefficients.

3.1. If $y{:}z < F/G$ then $z{:}y < G/F$

3.2. If $y{:}z < f_i$, $i = 1, \dots, k$, then $\left(\sum_{i=1}^{k} y_i\right){:}z < \sum_{i=1}^{k} f_i$

3.3. If $y{:}z < f$, $z{:}w < g$ then $y{:}w < fg$

Of these, 3.1. follows immediately from the definition while 3.2. and 3.3. can be verified without difficulty if the f_i, and f and g, are polynomials. However, 3.2. and 3.3. are not true unrestrictedly, as is shown by the following examples.

In 3.2, let $k = 2$, $y_1 = 1$, $y_2 = x + e^{-x}$, $z = 1 + x$, then $y_1{:}z < D/(D+1)$ since $y_1 = Dx$, $z = (D+1)x$, and $y_2{:}z < 1/(D+1)$ since $y_2 = 1(x + e^{-x})$, $z = (D+1)(x + e^{-x})$. However $(y_1 + y_2) : z = (1 + x + e^{-x}){:}(1 + x)$ does not belong to $[D/(D+1)] + [1/(D+1)] = 1$, since this would imply $1 + x + e^{-x} = 1 + x$.

Next, let $y = 1 + x$, $z = 1$, $w = 1 + x$ in 3.3. Then $y{:}z < (D+1)/D$ since $y = (D+1)x$, $z = Dx$, and $z{:}w < D$ since $z = D(1+x)$. However, $y{:}w$ does not belong to $[(D+1)/D].D = D + 1$, since $1 + x \neq (D+1)(1+x) = 2 + x$.

We shall now establish that 3.2 and 3.3 do hold under suitable restrictions.

3.4. *Theorem.* Let $f_i = G_i/F_i$ in 3.2 and let F be a common multiple of the F_i, $F = H_iF_i$, $i = 1, \dots, k$. Thus, $f = G/F$ where $G = H_1G_1 + \dots + H_kG_k$. Then the implication 3.2 holds provided F and G have no root in common.

Proof. Suppose first that $G \neq 0$. The conditions of the theorem imply the existence of functions $v_i \in C^n$ such that $y_i = H_iG_iv_i$,

$z = H_iF_iv_i = Fv_i$, $i = 1, \ldots, k$. Let v be an arbitrary but fixed solution of the equation $z = Fv$, then the functions $w_i = v_i - v$ satisfy $Fw_i = 0$ and

$$y_i = H_iG_iv_i = H_iG_iv + H_iG_iw_i$$

and so

$$\sum_{i=1}^{k} y_i = \left(\sum_{i=1}^{k} H_iG_i\right)v + \sum_{i=1}^{k} H_iG_iw_i = Gv + w$$

where $w = \sum_{i=1}^{k} H_iG_iw_i$ satisfies

$$Fw = F\left(\sum_{i=1}^{k} H_iG_iw_i\right) = \sum_{i=1}^{k} H_iG_iFw_i = 0$$

Now determine a function w_0 which satisfies

$$Gw_0 = w, \qquad Fw_0 = 0$$

Such a function exists by Lemma 2.1. Put $v_0 = v + w_0$, then $\sum_{i=1}^{k} y_i = Gv_0$, $z = Fv_0$, $\left(\sum_{i=1}^{k} y_i\right):z < G/F$, as required.

Next, we consider the possibility that $G = 0$. Since F and G have no roots in common, it follows that $F = c \neq 0$ and hence $F_i = c_i \neq 0$, where $c, c_i \in J$, and further $H_i = c/c_i$. Then there exist v_i such that $y_i = G_iv_i, z = F_iv_i = c_iv_i$ and so $v_i = (1/c_i)z$ and $y_i = (1/c_i)G_iz$. Since $(1/c_i)G_i \in J[D]$ it follows that

$$\sum_{i=1}^{k} y_i = \sum_{i=1}^{k} \frac{1}{c_i} G_iz = \frac{1}{c} Gz = 0$$

and so

$$\left(\sum_{i=1}^{k} y_i\right):z = 0:z < \frac{0}{F} = \frac{G}{F}$$

This completes the proof of 3.4.

Next we establish

3.5. *Theorem.* Let $f = F/H$, $g = G/K$. Then the implication of 3.3. applies provided H and G have no root in common.

Proof. By assumption, there exist functions y', z' such that $y = Fy'$, $z = Hy'$, $z = Gz'$, $w = Kz'$. Choose w' such that $y' = Gw'$, and try to determine $v = w' + v'$ so that $z' = Hv = Hw' + Hv'$ and at the same time $y' = Gv = Gw' + Gv'$. This requires that the two conditions

3.6. $$Gv' = 0$$

and $$Hv' = z' - Hw'$$

be satisfied simultaneously. But

$$G(z' - Hw') = Gz' - HGw' = z - Hy' = 0$$

and so the existence of a function v' as required is ensured by the lemma 2.1. Then $y = FGv$, $w = HKv$ and so $y:w < FG/HG$, proving the theorem.

4. *Application.* In the terminology of the present paper, the determination of a particular solution of a linear differential equation with constant coefficients amounts to the following. Given $z \in C^0$ and $F \in J[D]$, find y such that $z:y < F$. By 3.1, this condition is equivalent to $y:z < 1/F$. We now decompose $1/F$ into partial fractions G_i/F_i and find solutions y_i to the equations $y_i:z < G_i/F_i$. Then

$$(\Sigma y_i):z < \Sigma \frac{G_i}{F_i} = F$$

by 3.2, so that $y = \sum y_i$ is a solution of the problem. This is the standard procedure of the D calculus. When the decomposition into partial fractions is carried out in the complex domain each F_i/G_i is of the form $1/(D-\mu)^m$, and so (by 3.3, or otherwise) the y_i can be found by the repeated solution of equations of the form $y:z < 1/(D-\mu)$. If the discussion is restricted to the real domain, additional partial fractions of the form $g = \alpha D + \beta/(D^2 + \gamma D + \delta)^m$, $\gamma^2 - 4\delta < 0$, may appear, and the solution of the corresponding equations, $y:z < g$ may then be reduced to the solution of equations of the form

$$y:z < \frac{1}{D^2 + \gamma D + \delta}.$$

According to our approach the expressions G/F are not operators but ordinary polynomials. However, operators do appear implicitly since the pairs $y:z$ which correspond to any given G/F constitute a many-valued operator. In order to apply the values of the operator for given z we first find an integral w of the equation $Fw = z$ and then apply G to w. If the fraction $G/F \neq 0$ is reduced then it can be shown that all values of the operator for given z are obtained from one of them by adding arbitrary solutions of the homogeneous equation $Fw = 0$.

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MATHEMATICAL NOTES

2951. Lyness' periodic sequence

Several years ago, while working on number theory, R. C. Lyness observed that the recursion equation

$$a_n = (1 + a_{n-1})/a_{n-2}$$

led to a sequence with period 5; whatever values were taken for a_1 and a_2, one found $a_{n+5} = a_n$. He felt that this reflected some fundamental algebraic theorem, but could not find the background of this result. His intuition was perfectly correct. His result is related to the birational representation of the symmetric group (Burnside and E. H. Moore; see Messenger of Mathematics, XXX, p. 148, 1900–1901.) The result can be derived as follows. Let $-a_1$ equal the cross ratio $(ADBC)$. Let a_2, a_3, a_4, a_5 be obtained from a_1 by repeated cyclic permutation of A, B, C, D, E. Thus $-a_2 = (BECD)$ and $-a_3 = (CADE)$. All the cross ratios being invariant under projective transformation, there is no loss of generality in supposing A, B, C to have co-ordinates $0, \infty, 1$. Eliminating the co-ordinates of D, E we find

$$a_3 = (1 + a_2)/a_1$$

Four similar equations, obtained by cyclic permutation of 1, 2, 3, 4, 5, also hold. By suitable choice of D, E we can make a_1 and a_2 take any values we like. Lyness' result is thus proved.

W. W. Sawyer

2952. Cycles

The method of proof given in Professor Sawyer's note, unlike the simple proof by calculating successive terms from an arbitrary first pair:

$$x, y, (1 + y)/x, (1 + x + y)/xy, (1 + x)/y, x, y\ldots \qquad \text{(i)}$$

and seeing 'that it works,' can be generalised and leads to the following system of cycles:

$$\begin{array}{lll}
u_2 + (u_1 - 1) & = 0 & 2 = \text{4-cycle} \\
u_3u_1 + (u_2 - 1) & = 0 & \text{5-cycle} \\
u_4(u_2 + u_1 - 1) + (u_3 - 1)u_1 & = 0 & \text{6-cycle} \\
u_5(u_3u_1 + u_2 - 1) + (u_4 - 1)(u_2 + u_1 - 1) & = 0 & \text{7-cycle} \\
u_6[u_4(u_2 + u_1 - 1) + u_1(u_3 - 1)] & & \\
\qquad + (u_5 - 1)(u_3u_1 + u_2 - 1) & = 0 & \text{8-cycle}
\end{array}$$

et cetera

where u stands for $-a_{n+}$, e.g. $u_3 = -a_{n+3}$. For a t-cycle arbitrary $a_1, a_2, \ldots, a_{t-3}$ are taken and subsequent a calculated from the recurrence relation. One finds that, for all integral n, $a_{n+t} = a_n$.

Burnside in his article, for the reference to which I am most grateful to Professor Sawyer, proves that the substitution B on the $(n-3)$ numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n-3}$ given by

$$B : \alpha_1' = \alpha_2, \alpha_2' = \alpha_3, \ldots, \alpha_{n-4}' = \alpha_{n-3},$$

$$\alpha_{n-3}' = -1 + \frac{\alpha_{n-3}}{-1+} \frac{\alpha_{n-4}}{-1+} \cdots \frac{\alpha_1}{-1} \tag{ii}$$

generates a cyclic group. $-\alpha_1$ is the cross-ratio (x_1, x_2, x_4, x_3) and the remaining $-\alpha_r$ are the results of applying S^{r-1} to this cross ratio, where S is the cyclic permutation $(1, 2, 3, 4, \ldots, n)$. x_r, $r = 1$ to n, can be chosen so that α_r, $r = 1$ to $n-3$, have arbitrary values, but α_{n-2}, α_{n-1} and α_n are then fixed.

Burnside's proof depends on the identity

$$-(x, y, a, z) = -1 + (w, x, z, y)/(w, x, a, y)$$

which is easily verifiable.

Put $$t_r = -(x_r, x_{r+1}, x_1, x_{r+2}).$$

Then the identity gives, with $w = x_{r-1}$,

$$t_r = -1 + \frac{\alpha_{r-1}}{t_{r-1}}, \tag{iii}$$

and

$$\begin{aligned} \alpha_{n-2} &= -S^{n-3}(x_1, x_2, x_4, x_3) \\ &= -(x_{n-2}, x_{n-1}, x_1, x_n) \\ &= t_{n-2} \\ &= -1 + \frac{\alpha_{n-3}}{t_{n-3}} \\ &= -1 + \frac{\alpha_{n-3}}{-1+} \frac{\alpha_{n-4}}{t_{n-4}}. \end{aligned}$$

Since $t_1 = -1$, repeated use of (iii) makes α_{n-2} equal to the continued fraction on the right side of (ii) Thus the substitution B on the numbers α_r, $r = 1$ to $n-3$, changes them to α_{r+1}, $r = 1$ to $n-3$. Since $\alpha_{n+t} = \alpha_t$, B generates a cyclic group of order n.

The cycles given at the beginning of this note were found much less elegantly than in Burnside's proof and are equivalent to (ii) with α_{n-2} written for α_{n-3}' and then u written for $1 + \alpha$.

It may be of interest to add that the original 5-cycle came from the problem of finding three integers a, b, c such that the sum and the difference of each pair are squares. From

$$b + c = p^2 \qquad c + a = q^2 \qquad a + b = r^2$$

$$c - b = u^2 \qquad c - a = v^2 \qquad b - a = w^2$$

it follows that

$$p^2 - q^2 = w^2, \qquad q^2 - r^2 = u^2, \qquad p^2 - r^2 = v^2 \tag{iv}$$

It was noticed by Mr. D. F. Ferguson (see note 1847) that if p_1 q_1, r_1, u_1, v_1, w_1 is a solution of (iv), then so is

$$p_2 = p_1v_1, q_2 = p_1w_1, r_2 = v_1w_1, u_2 = r_1w_1, v_2 = q_1v_1, w_2 = p_1u_1$$

Writing $\alpha_1 = q$, $\alpha_2 = r$, $\alpha_3 = u$, $\alpha_4 = v$, $\alpha_5 = w$ and putting $p = 1$ so as to have a substitution on five numbers instead of on the ratio of six, Ferguson's substitution is equivalent to

$$T\colon \alpha_1' = \alpha_5/\alpha_4, \alpha_2' = \alpha_5, \alpha_3' = \alpha_2\alpha_5/\alpha_4, \alpha_4' = \alpha_1, \alpha_5' = \alpha_3/\alpha_4$$

The table we get from repeating this substitution is

	1	2	3	4	5
I	q	r	u	v	w
T	w/v	w	rw/v	q	u/v
T^2	u/qv	u/v	uw/qv	w/v	rw/qv
T^3	r/q	rw/qv	ru/qv	u/qv	u/q
T^4	v	u/q	rw/q	r/q	r
T^5	q	r	u	v	w

The similarity in the five terms of columns 4 and 1 to the original terms (i) of the 5-cycle is apparent. With $w = q + 1$, $u = v + q + 1$ and $r = v + 1$ we can write $\beta_1 = \alpha_4$ and $\beta_2 = \alpha_1$ to get the substitution U, on two numbers,

$$U\colon \beta_1' = \beta_2, \quad \beta_2' = (\beta_2 + 1)/\beta_1.$$

which is the original 5-cycle.

R. C. Lyness

2953. Vector potential

It has been suggested to me that the formula

$$\mathbf{g} = -\mathbf{r}\wedge\int_0^1 \mathbf{f}(\lambda\mathbf{r})\lambda\, d\lambda,$$

whenever the integral exists, for a vector potential of the solenoidal

field div $\mathbf{f} = 0$, and due to H. Liebmann, is not as well known as it deserves to be. It is included in L. Brand's text, *Vector Analysis*, 1957, being an extract from a more complete paper by the same writer in the *American Mathematical Monthly*, vol. 57, 1950, pp. 161–167.

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2954. On note 2889

I am doubtful whether a proof that makes use of an arbitrary orthogonal triad **i, j, k** can properly be described as "a proof using only vector methods." Here is an alternative and shorter proof of Durand's relation.

Write Stokes' theorem in the form

$$\oint \mathbf{dr} \cdot \mathbf{v} = \int (\mathbf{dS} \times \nabla) \cdot \mathbf{v}$$

and replace $\mathbf{v}$ by $\mathbf{f} \times \mathbf{t}$, where $\mathbf{t}$ is an arbitrary constant vector. By taking $\mathbf{t}\cdot$ outside the integral signs one finds

$$\begin{aligned}\oint \mathbf{dr} \times \mathbf{f} &= \int (\mathbf{dS} \times \nabla) \times \mathbf{f}\\ &= \int \mathbf{dS} \times (\nabla \times \mathbf{f}) + \int \nabla \times (\mathbf{f} \times \mathbf{dS})\\ &= \int \mathbf{dS} \times (\nabla \times \mathbf{f}) + \int (\mathbf{dS} \cdot \nabla)\mathbf{f} - \int \mathbf{dS}(\nabla \cdot \mathbf{f}),\end{aligned}$$

which is the desired result. It is to be understood that the operator ∇ here operates only on $\mathbf{f}$, not on $\mathbf{dS}$.

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Editorial Note. Proofs of this result were also received from E. Crosby, P. Morley, E. B. Spratt and K. L. Wardle.

2955. Centroid of a uniform circular arc

Given a uniform circular arc of radius R and subtending an angle 2α at the centre O.

It is evident that the centroid G must lie on the bisector of the angle 2α and that:

$$OG = R f(\alpha) \tag{1}$$

It is equally evident that if we divide the arc into two equal parts AB and AC that the centroids G_1 and G_1' are collinear with G and that G_1G_1' is perpendicular to OA. Further:

$$OG_1 = R f(\alpha/2) \tag{2}$$

and
$$OG = OG_1 \cos \alpha/2 \tag{3}$$

Whence
$$f(\alpha) = f(\alpha/2) \cos \alpha/2$$

Similarly $$f(\alpha/2) = f(\alpha/4)\cos\alpha/4$$

Whence $$f(\alpha) = \cos\frac{\alpha}{2}\cos\frac{\alpha}{4}\dots\cos\frac{\alpha}{2^n}f\left(\frac{\alpha}{2^n}\right)$$

Now $\lim_{\alpha\to 0} f(\alpha) = 1$, since G must lie between chord and arc, and

$$\frac{\sin\alpha}{\alpha} = \frac{2\sin\alpha/2\cos\alpha/2}{\alpha} = \cos\frac{\alpha}{2}\,\frac{\sin\alpha/2}{\alpha/2}$$

$$\frac{\sin\alpha}{\alpha} = \cos\frac{\alpha}{2}\cos\frac{\alpha}{4}\dots\cos\frac{\alpha}{2^n}\cdot\frac{\sin\alpha/2^n}{\alpha/2^n} \text{ (Euler).}$$

So that we find:

$$OG = R\frac{\sin\alpha}{\alpha}.$$

University of Queensland, Australia E. T. STELLER

2956. The distributive law of indices

Solve $(8-x)^{2/3} + (27+x)^{2/3} = (8-x)^{1/3}(27+x)^{1/3} + 7$.

We have

$$(8-x)^{2/3} + (27+x)^{2/3} = (216 - 19x - x^2)^{1/3} + 7$$

and so

$$8^{2/3} - x^{2/3} + 27^{2/3} + x^{2/3} = 216^{1/3} - 19^{1/3}x^{1/3} - x^{2/3} + 7.$$

Hence

$$4 + 9 = 6 - 19^{1/3}x^{1/3} - x^{2/3} + 7,$$

that is

$$19^{1/3}x^{1/3} + x^{2/3} = 0,\quad 19x + x^2 = 0$$

and so $x = 0$ or -19. A more conventional approach shows that these are indeed the correct roots.

E. M. WRIGHT

2957. An inequality of Schur's type for five variables

THEOREM: *If x, y, z, u, v are all positive, or all negative, then*

$$\begin{aligned}(x-y)(x-z)(x-u)(x-v) &+ (y-x)(y-z)(y-u)(y-v)\\ &+ (z-x)(z-y)(z-u)(z-v)\\ &+ (u-x)(u-y)(u-z)(u-v)\\ &+ (v-x)(v-y)(v-z)(v-u)\\ &> 0 \qquad (1)\end{aligned}$$

Suppose x, y, z, u, v are all positive. Since both sides of (1) are symmetric in x, y, z, u, v there is no loss of generality in assuming $x > y > z > u > v$.

Consider the first and second terms of (1). Clearly the first is positive and the second negative, thus

$$(x-y)[(x-z)(x-u)(x-v)-(y-z)(y-u)(y-v)]$$

is positive, since $x > y$ etc. (2)

The third term is positive. (3)

Consider the fourth and fifth terms:

$$(u-v)[(x-v)(y-v)(z-v)-(x-u)(y-u)(z-u)]$$

is also positive, since $u > v$ etc. (4)

From (2), (3) and (4), the inequality follows.

Now suppose x, y, z, u, v are all negative, then we can write

$$x=-\alpha, y=-\beta, z=-\gamma, u=-\delta, v=-\epsilon$$

where $\alpha, \beta, \gamma, \delta, \epsilon$ are all positive and (1) becomes

$$\begin{aligned}(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\epsilon)&+(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\epsilon)\\&+(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\epsilon)\\&+(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\epsilon)\\&+(\epsilon-\alpha)(\epsilon-\beta)(\epsilon-\gamma)(\epsilon-\delta)\\&>0 \qquad (5)\end{aligned}$$

the negative signs being absorbed as there are even number of factors in each term. The inequality (5) is of the same type as (1) and hence the inequality follows at once.

Particular cases can be deduced:

Suppose in (1) we put $v = 0$, then

$$\begin{aligned}x(x-y)(x-z)(x-u)&+y(y-x)(y-z)(y-u)\\&+z(z-x)(z-y)(z-u)\\&+u(u-x)(u-y)(u-z)\\&+xyzu>0 \qquad (6)\end{aligned}$$

Further supposing u is also zero we get

$$x^2(x-y)(x-z)+y^2(y-x)(y-z)+z^2(z-x)(z-y)>0 \quad (7)$$

The result (7) is a particular case of the inequality

$$x^n(x-y)(x-z)+y^n(y-x)(y-z)+z^n(z-x)(z-y)>0 \quad (8)$$

The result (8) is proved for $n > 0$ and $n < 1$. (Barnard and Child; *Higher Algebra*; p. 217).

It seems to be a difficult problem to extend the inequality to seven or more odd number of variables. The result is not true for four variables as would appear from (6).

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2958. Two tournament problems

(1) In the February, 1960, issue of the Mathematical Gazette Dr. Garreau refers to the following Cambridge Scholarship question as one which has occasioned many enquiries to the Problems Bureau:

'A tennis match is played between two teams, each player playing one or more members of the other team. Further, (i) any two members of the same team have exactly one opponent in common, (ii) no two members belonging to the same team play all the members of the other team between them.

Prove that two players who do not play each other have the same number of opponents. Deduce that any two players, whether belonging to the same team or different teams, have the same number of opponents.'

It can also be shown that if it is possible for a player to have n opponents then each team has $n^2 - n + 1$ members. When n is 1 or 2 this is easily found to be impossible but for $n = 3$ solutions do exist. Thus if the members of teams A and B are a_1 to a_7 and b_1 to b_7 respectively, and if $a_i b_j$ represents a game played between a_i and b_j, then one solution is made up of the games

$$\begin{array}{ccccccc} a_1b_1 & a_2b_1 & a_3b_2 & a_4b_3 & a_5b_2 & a_6b_3 & a_7b_1 \\ a_1b_2 & a_2b_4 & a_3b_4 & a_4b_4 & a_5b_5 & a_6b_5 & a_7b_6 \\ a_1b_3 & a_2b_5 & a_3b_6 & a_4b_7 & a_5b_7 & a_6b_6 & a_7b_7. \end{array}$$

All other solutions for $n = 3$ may be obtained from this one by permutation of the members of the teams.

(2) A more useful and probably more familiar problem is that of pairing opponents in a competition involving $2n$ players each of whom is to play exactly one game against each of the other players. There are thus $n(2n - 1)$ games to be played and, if each day there are at most n games played with no player playing in more than one of them, it is required to organize the games so that the contest is completed in $2n - 1$ days. For particular values of n solutions to this problem are familiar to the organizers of chess contests but do not appear to be widely known and it might be worth noting the following solution for general $n > 2$.

The day on which the i^{th} and j^{th} players compete against one another is

$$
\begin{array}{lll}
i+j-2 & \text{if} & i+j < 2n+1; \\
i+j-2n-1 & \text{if} & 3 < i < 2n-1, \quad 3 < j < 2n-1 \\
 & & \text{and} \quad i+j > 2n+2; \\
2j-2 & \text{if} & i = 2n \quad \text{and} \quad 2 < j < n; \\
2j-2n-1 & \text{if} & i = 2n \quad \text{and} \quad n+1 < j < 2n-1; \\
2i-2 & \text{if} & 2 < i < n \quad \text{and} \quad j = 2n; \\
2i-2n-1 & \text{if} & n+1 < i < 2n-1 \quad \text{and} \quad j = 2n.
\end{array}
$$

This may be represented by a matrix of order $2n$ in which the element in position (i, j) represents the day on which the i^{th} and j^{th} players meet, elements in the main diagonal being ignored. For $n = 6$ the appearance of the matrix indicates quite clearly the way in which the solution may be written down for any $n > 2$.

–	1	2	3	4	5	6	7	8	9	10	11
1	–	3	4	5	6	7	8	9	10	11	2
2	3	–	5	6	7	8	9	10	11	1	4
3	4	5	–	7	8	9	10	11	1	2	6
4	5	6	7	–	9	10	11	1	2	3	8
5	6	7	8	9	–	11	1	2	3	4	10
6	7	8	9	10	11	–	2	3	4	5	1
7	8	9	10	11	1	2	–	4	5	6	3
8	9	10	11	1	2	3	4	–	6	7	5
9	10	11	1	2	3	4	5	6	–	8	7
10	11	1	2	3	4	5	6	7	8	–	9
11	2	4	6	8	10	1	3	5	7	9	–

For $n = 2$ this general form of matrix provides the only solution apart from permutations among the players, but for $n > 3$ other solutions are possible.

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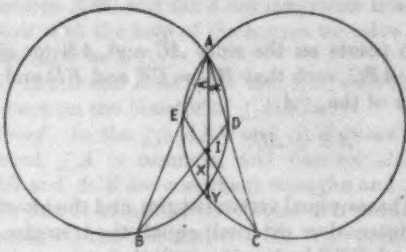
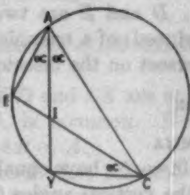
2959. The "equal bisector" theorem

This, as usually stated is: If in a triangle ABC the bisectors of the angles B and C are equal then $AB = AC$.

However it is not necessary that the equal lines should bisect the angles B and C; all that is needed is that they should intersect on the bisector of the angle A.

Given: $BD = EC$, BD and EC meet on AI the bisector of A. Draw the circles ADB, AEC. These are equal since the equal chords BD, CE subtend the same angle (2α) at their circumferences.





Let AI cut these circles in X and Y.

It will be shown that X and Y are coincident, and the rest is elementary.

XD, XB, YC and YE are all equal—subtending α at the circumferences of equal circles.

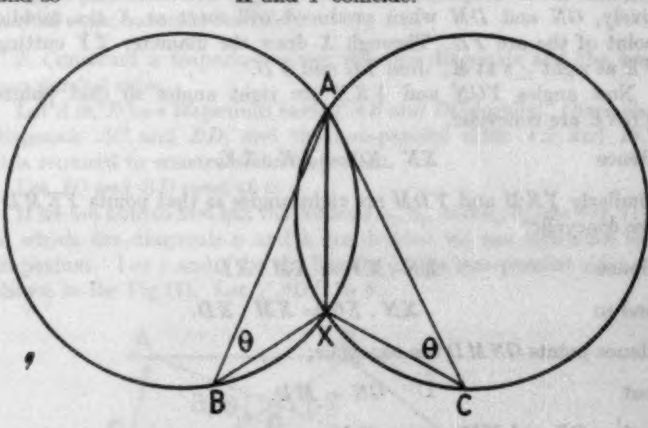
Therefore $\quad ECY = EAY$ (same segment) $= CAY$,

and so $\quad YC$ is a tangent at C to the circle AIC

Whence $\quad YC^2 = YI \,.\, YA$. Similarly $XD^2 = XI \,.\, XA$

so that $\quad XI \,.\, XA = YI \,.\, YA$

and so $\quad X$ and Y coincide.



(Of many continuations this is as good as any:—

The angles θ are equal, subtended by the common chord.

Therefore $\quad AXB = AXC$ and so $AB = AC$.)

L. J. Russell

2960. Two problems

1. D and E are two points on the sides AC and AB (or sides produced) of a triangle ABC, such that $BD = CE$ and BD and CE intersect on the bisector of the $\angle A$.

Then $$AB = AC.$$

LEMMA

If two $\triangle$s have equal bases, equal vertical angles, and the bisectors of the vertical angles (internal or external) equal, the triangles are congruent. Let ABC and DEF be two $\triangle$s such that

$$BC = EF; \quad \angle BAC = \angle EDF;$$

and $$\text{bisector} AL = \text{bisector } DM.$$

It is required to prove that the two $\triangle$s ABC and DEF are congruent.

Proof. Apply $\triangle ABC$ to $\triangle DEF$ such that BC coincides with EF and A falls on the same side of EF as D. Let GFE be the new position of the $\triangle ABC$ and GN the position of the bisector AL.

Now $$\angle FGE = \angle BAC = \angle EDF$$

and so the points $FGDE$ are concyclic. Complete the $\odot FGDE$.

Since GN and DM are bisectors of the $\angle$s FGE and EDF respectively, GN and DM when produced will meet at X the middle point of the arc FE. Through X draw the diameter XY cutting FE at right $\angle$s at K. Join YG and YD.

Now angles YGN and YKN are right angles so that points $YGNK$ are concyclic.

Hence $$XN \,.\, XG = XK \,.\, XY.$$

Similarly YKM and YDM are right angles so that points $YKMD$ are concyclic.

Hence $$XK \,.\, XY = XM \,.\, XD.$$

and so $$XN \,.\, XG = XM \,.\, XD.$$

Hence points $GNMD$ are concyclic;

but $$GN = MD.$$

and so GD and NM are parallel.

$GFED$ is concyclic and GD and FE are parallel

so that $$GF = DE \text{ and } GE = DF.$$

Hence GFE and DEF are congruent triangles.

Therefore ABC and DEF are congruent triangles.

Now with the help of the lemma we solve problem 1.

Let ABC be a $\triangle$.

(i) Let D and E be in AC and AB, such that $BD = CE$ and they intersect on the bisector of $\angle BAC$ at O.

Proof. In the $\triangle$s ABD and ACE bases BD and CE are equal, vertical $\angle A$ is common, and bisector AO, is common. Hence ABD and ACE are congruent triangles and $AB = AC$.

(ii) Let D and E be in AC and AB produced. Let BD and CE meet at O_2 the bisector of $\angle A$. The proof is the same as in (i).

(iii) Let D and E be in CA and BA produced. Let BD and CE when produced meet the bisector at O_3.

In the triangles BAD and CAD,

$$BD = CE,$$

$$\angle BAD = \angle CAE,$$

and the external bisector AO_3 is common to both. Hence BAD and CAD are congruent triangles and

therefore $$BA = CA.$$

Note: The well known converse that if the bisectors of the base angles terminated by the sides are equal the triangle is isosceles is only a particular case of problem 1. A similar proof deals with the case of equal external bisectors.

2. Construct a trapezium given the two diagonals and the two non-parallel sides.

Let $ABCD$ be a trapezium having AB and DC parallel. Given the diagonals AC and BD, and the non-parallel sides AD and BC, it is required to construct the trapezium.

Let AC and BD meet at O.

If we are able to find out the sections a_1, a_2, and b_1, b_2 (see Fig. (1)) in which the diagonals a and b are divided we can construct the trapezium. Let c and d be the lengths of the non-parallel sides as shown in the Fig (1). Let $\angle AOD$ be θ.

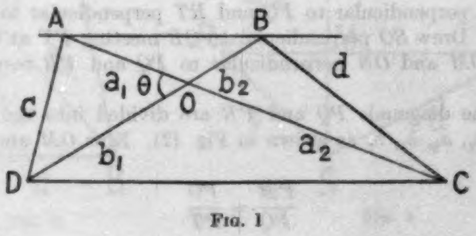


FIG. 1

Now AOB and COD are similar triangles.

so that $$\frac{a_1}{b_2} = \frac{a_2}{b_1}, \quad \text{or} \quad a_1b_1 = a_2b_2.$$

In $\triangle AOD$ $$c^2 = a_1^2 + b_1^2 - 2a_1b_1 \cos\theta;$$

and in the $\triangle BOC$
$$d^2 = a_2^2 + b_2^2 - 2a_2b_2 \cos\theta.$$

Hence
$$c^2 - d^2 = a_1^2 - a_2^2 + b_1^2 - b_2^2;$$

since
$$a_1b_1 = a_2b_2$$

therefore
$$c^2 - d^2 + a_2^2 - a_1^2 + b_2^2 - b_1^2 = 0.$$

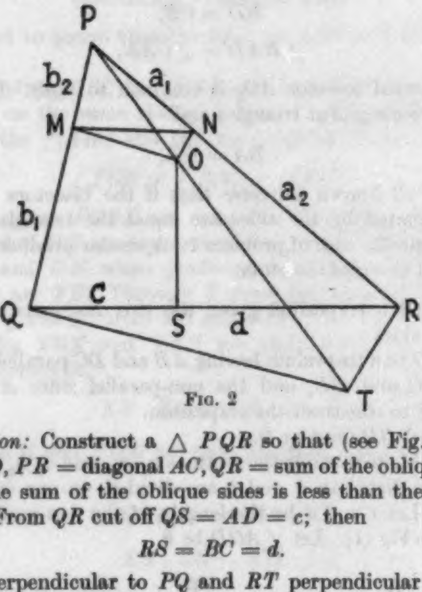


FIG. 2

Construction: Construct a $\triangle$ PQR so that (see Fig. (2)) PQ = diagonal BD, PR = diagonal AC, QR = sum of the oblique sides AD and BC; the sum of the oblique sides is less than the sum of the diagonals. From QR cut off $QS = AD = c$; then

$$RS = BC = d.$$

Draw QT perpendicular to PQ and RT perpendicular to PR and join PT. Draw SO perpendicular to QR meeting PT at O. From O draw OM and ON perpendicular to PQ and PR respectively. Join MN.

Now the diagonals PQ and PR are divided into the required sections a_1, a_2, b_1, b_2 as shown in Fig. (2). Now OM and TQ are parallel

so that $$\frac{PM}{PQ} = \frac{PO}{PT}$$

and since ON and TR are parallel;

$$\frac{PO}{PT} = \frac{PN}{PR}.$$

Hence

$$\frac{PM}{PQ} = \frac{PN}{PR}$$

and so MN is parallel to QR whence

$$a_1b_1 = a_2b_2.$$

Since OM, ON and OS are perpendiculars to the sides of the $\triangle$ (Fig. 2)

$$c^2 - d^2 + a_2^2 - a_1^2 + b_2^2 - b_1^2 = 0.$$

Hence these are the required sections of the diagonal.

Construct a $\triangle AOB$ with sides c, a, and b, (see Fig. 3). Produce AO to C making $OC = a_2$. Produce BO to D making $OD = b_2$ and join CD, CB and DA. Now $ABCD$ is the required trapezium. Since

$$\frac{a_1}{a_2} = \frac{b_2}{b_1}$$

AD and BC are parallel.

Since $ABCD$ is a trapezium

$$a_1^2 + b_1^2 - c^2 = a_2^2 + b_2^2 - CD^2$$
$$c^2 - CD^2 + a_2^2 - a_1^2 + b_2^2 - b_1^2 = 0.$$

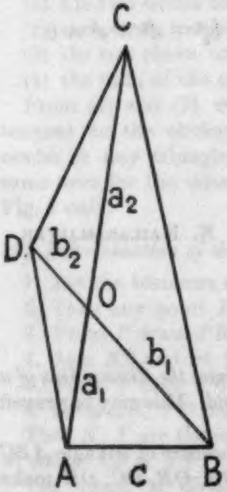


FIG. 3

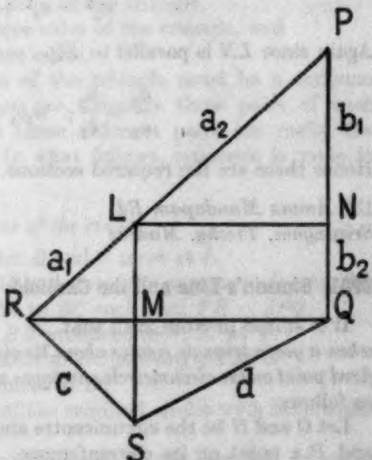


FIG. 4

But $\quad c^2 - d^2 + a_2^2 - a_1^2 + b_2^2 - b_1^2 = 0.$ (proved)

therefore $\quad CD^2 = d^2 \quad \text{or} \quad CD = d.$

Hence $ABCD$ is the required trapezium.

Second Method

Draw a right angled triangle PQR (Fig. 4) such that the hypotenuse PR is the longer diagonal and PQ is another diagonal; on QR construct $\triangle QRS$ with the oblique sides of the trapezium as sides. Let

$$RS = AB = C$$

and

$$QS = CD = d.$$

Through S draw SM perpendicular to QR meeting PR at L. Draw LN perpendicular to PQ.

Now the two diagonals PR and PQ are divided into the required sections.

In the $\triangle PQR$ since LM and LN are perpendicular,

$$a_1^2 - a_2^2 + b_1^2 - b_2^2 + QM^2 - MR^2 = 0.$$

Since SM is perpendicular to QR

$$QM^2 - MR^2 = d^2 - c^2,$$

$$a_1^2 - a_2^2 + b_1^2 - b_2^2 + d^2 - c^2 = 0.$$

Again since LN is parallel to RQ.

$$\frac{a_1}{a_2} = \frac{b_2}{b_1}; \quad a_1b_1 = a_2b_2.$$

Hence these are the required sections.

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2961. Simson's Line and the Cardioid

It is stated in Note 2915 that *when a given triangle rotates about its circumcentre the Simson line of a fixed point on the circumcircle envelopes a cardioid.* This may be proved as follows:

Let O and H be the circumcentre and orthocentre of triangle ABC and P a point on its circumference. Let OA, OB, OC, OH make angles α, β, γ, η with OP, measured anticlockwise from OP. Then

the Simson line of P bisects PH and makes an angle $\frac{1}{2}(\alpha + \beta + \gamma)$ with OP (as proved in Note 2338, Vol. XXXVII, p. 320). Draw YHX parallel to the Simson line. Draw the circle, centre O and radius OH, and let KOL be a diameter of it such that angle $KOP = \alpha + \beta + \gamma - 3\eta$, measured anticlockwise. Then the turn from LH to $OH = \frac{1}{2}(\text{angle } KOP + \eta), = \frac{1}{2}(\alpha + \beta + \gamma) - \eta, =$ the turn from OH to HY. Thus OH bisects angle LHY.

As the triangle rotates, the angles $\alpha, \beta, \gamma, \eta$ all increase by the same amount, and the angle KOP therefore remains constant. L is thus a fixed point on the circle KHL on which H moves, and the envelope of HY is the caustic of that circle with radiant point L, i.e. a cardioid having its cusp on OK, the radius of its base-circle being $\frac{1}{3}OH$. The envelope of the Simson line is, by similarity, a cardioid with its cusp-line bisecting OP and parallel to OK, the radius of its base-circle being $\frac{1}{6}OH$.

The second theorem in Note 2915 may be proved on similar lines, the angle KOP being now increased by twice the constant angle mentioned.

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2962. Equal circles in a triangle

Suppose we have a triangle and two circles with the following properties:

(1) the two circles touch each other externally,
(2) each circle touches two sides of the triangle,
(3) the two circles touch three sides of the triangle, and
(4) the radii of the circles are equal.

From (2) and (3), one side of the triangle must be a common tangent to the circles. There are therefore three pairs of such circles in any triangle. The three different pairs are really the same save for the notation. In what follows, reference is made to Fig. 1 only.

(A) *Determination of the centres of the circles*

1. Let the bisectors of angles B and C meet at I.
2. Take any point P on BI and drop PQ perpendicular to BC.
3. From P draw PR parallel to BC such that $PR = 2PQ$.
4. Join BR and let BR or BR produced meet IC at Y.
5. From Y draw a line parallel to BC meeting BI at X.
6. Drop XE, YF perpendicular to BC.

Then X, Y are the centres of the required circles with radii equal to either XE or YF.

The proof is obvious on considering the similar triangles concerned.

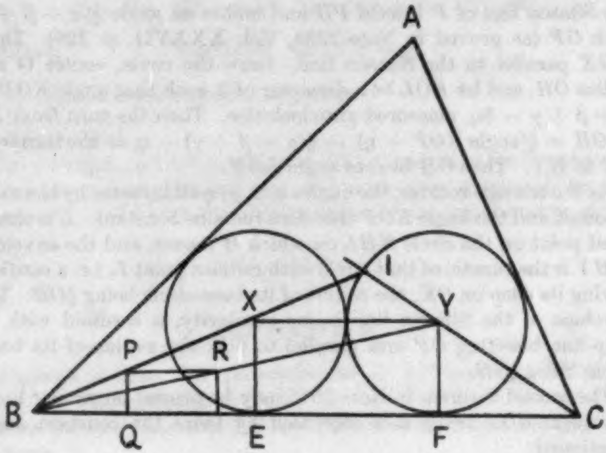

FIG. 1

(B) *Length of radii of the circles*

With the usual notations for the elements and various radii of the circles of the triangle and the formula: $r = 4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$, let r_a be the length of the radii of the equal circles with "a" as their common tangent.

From triangles BXE and CYF,

$$BE = r_a \cot \tfrac{1}{2}B; \qquad CF = r_a \cot \tfrac{1}{2}C.$$

From rectangle $XYFE$, $EF = 2r_a$.

Hence $$2r_a + r_a \cot \tfrac{1}{2}B + r_a \cot \tfrac{1}{2}C = a$$

or $$r_a = \frac{a}{2 + \cot \frac{1}{2}B + \cot \frac{1}{2}C}$$

$$\frac{1}{r_a} = \frac{2}{a} + \frac{1}{a}(\cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C)$$

$$= \frac{2}{a} + \frac{1}{a}\,\frac{\cos \frac{1}{2}B \sin \frac{1}{2}C + \cos \frac{1}{2}C \sin \frac{1}{2}B}{\sin \frac{1}{2}B \sin \frac{1}{2}C}$$

$$= \frac{2}{a} + \frac{1}{2R \sin A}\,\frac{\sin \frac{1}{2}(B + C)}{\sin \frac{1}{2}B \sin \frac{1}{2}C}$$

$$= \frac{2}{a} + \frac{1}{4R \sin \frac{1}{2}A \cos \frac{1}{2}A}\,\frac{\cos \frac{1}{2}A}{\sin \frac{1}{2}B \sin \frac{1}{2}C}$$

$$= \frac{2}{a} + \frac{1}{4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C}$$

$$= \frac{2}{a} + \frac{1}{r}$$

The other sets of radii are respectively given by

$$\frac{1}{r_b} = \frac{2}{b} + \frac{1}{r}, \qquad \frac{1}{r_c} = \frac{2}{c} + \frac{1}{r}.$$

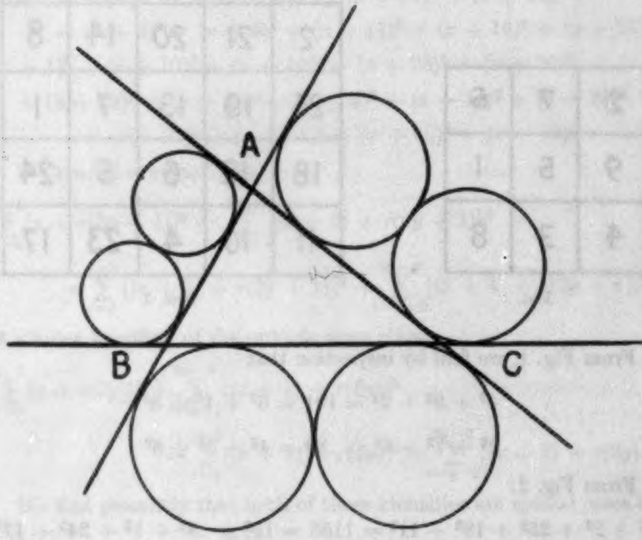

FIG. 2

A similar process may be operated on the escribed circles. It is found that the radii of the pairs of circles are respectively the reciprocals of

$$\frac{2}{a} + \frac{1}{r_1}, \quad \frac{2}{b} + \frac{1}{r_2}, \quad \frac{2}{c} + \frac{1}{r_3}.$$

with the usual interpretation for r_1, r_2, r_3. (Fig. 2)

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2963. The Magic of Squares

The purpose of this paper is to show that in many magic squares not only are the sums of the numbers in all columns, rows and two main diagonals equal, but also that the sums of the squares of the numbers in certain paired columns and rows are equal.

We shall deal here only with "outside" rows and columns.

Consider the following magic squares

2	7	6
9	5	1
4	3	8

Fig. 1

9	3	22	16	15
2	21	20	14	8
25	19	13	7	1
18	12	6	5	24
11	10	4	23	17

Fig. 2

From Fig. 1, we find by inspection that

$$4^2 + 9^2 + 2^2 = 101 = 6^2 + 1^2 + 8^2$$

$$2^2 + 7^2 + 6^2 = 89 = 4^2 + 3^2 + 8^2$$

From Fig. 2:

$$9^2 + 2^2 + 25^2 + 18^2 + 11^2 = 1155 = 15^2 + 8^2 + 1^2 + 24^2 + 17^2$$

$$9^2 + 3^2 + 22^2 + 16^2 + 15^2 = 1055 = 11^2 + 10^2 + 4^2 + 23^2 + 17^2$$

If we arrange the numbers in each of the four "outside" columns in order of magnitude, a pattern begins to emerge:

$$\begin{array}{ccccccccccc} 1^2 & + & 6^2 & + & 8^2 & = & 2^2 & + & 4^2 & + & 9^2 \\ & 5 & & 2 & & & & 2 & & 5 & \end{array}$$

$$\begin{array}{ccccccccccccccccccc} 1^2 & + & 8^2 & + & 15^2 & + & 17^2 & + & 24^2 & = & 2^2 & + & 9^2 & + & 11^2 & + & 18^2 & + & 25^2 \\ & 7 & & 7 & & 2 & & 7 & & & & 7 & & 2 & & 7 & & 7 & \end{array}$$

A quick check of the outside columns of a 7×7 square constructed in a similar way will enable us to say:

$$1^2 + 10^2 + 19^2 + 28^2 + 30^2 + 39^2 + 48^2$$

$$9 \quad 9 \quad 9 \quad 2 \quad 9 \quad 9$$

$$= 2^2 + 11^2 + 20^2 + 22^2 + 31^2 + 40^2 + 49^2$$

$$9 \quad 9 \quad 2 \quad 9 \quad 9 \quad 9$$

We can add any quantity x to each number:

$$(x+1)^2 + (x+6)^2 + (x+8)^2 = (x+2)^2 + (x+4)^2 + (x+9)^2$$

$$(x+1)^2 + (x+8)^2 + (x+15)^2 + (x+17)^2 + (x+24)^2$$
$$= (x+2)^2 + (x+9)^2 + (x+11)^2 + (x+18)^2 + (x+25)^2$$

$$(x+1)^2 + (x+10)^2 + (x+19)^2 + (x+28)^2 + (x+30)^2$$
$$+ (x+39)^2 + (x+48)^2 = (x+2)^2 + (x+11)^2 + (x+20)^2$$
$$+ (x+22)^2 + (x+31)^2 + (x+40)^2 + (x+49)^2$$

We are led eventually to

$$\sum_{r=0}^{y-1} \{x + r(2y+1)\}^2 + \sum_{r=y-1}^{2y-3} \{(x+2) + r(2y+1)\}^2$$
$$= \sum_{r=0}^{y-2} \{(x+1) + r(2y+1)\}^2 + \sum_{r=y-2}^{2y-3} \{(x+3) + r(2y+1)\}^2$$

A similar handling of the outside rows gives

$$\sum_{r=0}^{y-1} \{x + r(2y)\}^2 + \sum_{r=y-1}^{2y-3} \{(x+1) + r(2y)\}^2$$
$$= \sum_{r=0}^{y-2} \{(x+1) + r(2y)\}^2 + \sum_{r=y-2}^{2y-3} \{(x+2) + r(2y)\}^2$$

We find presently that both of these identities are special cases of

$$\sum_{r=0}^{y-1} \{x + r(2y+a)\}^2 + \sum_{r=y-1}^{2y-3} \{(x+a+1) + r(2y+a)\}^2$$
$$= \sum_{r=0}^{y-2} \{(x+1) + r(2y+a)\}^2 + \sum_{r=y-2}^{2y-3} \{(x+a+2) + r(2y+a)\}^2$$

We can now consider what happens when the first two terms no longer differ by unity.

We write down again

$$1^2 + 6^2 + 8^2 = 2^2 + 4^2 + 9^2$$

$$5 \quad 2 \qquad\qquad 2 \quad 5$$

The outside rows of Fig. 3 give us

$$3^2 + 11^2 + 13^2 = 5^2 + 7^2 + 15^2 \rightarrow 1^2 + 9^2 + 11^2 = 3^2 + 5^2 + 13^2$$

8 2 2 8

If we conjecture

$$1^2 + 12^2 + 14^2 = 4^2 + 6^2 + 17^2 \text{ and } 1^2 + 15^2 + 17^2 = 5^2 + 7^2 + 21^2,$$

11 2 2 11 14 2 2 14

we enable ourselves to construct Figs. 4 and 5!

3	13	11
17	9	1
7	5	15

FIG. 3

4	11	12
17	9	1
6	7	14

FIG. 4

5	13	15
21	11	1
7	9	17

FIG. 5

Our subsequent result turns out to be

$$x^2 + (x + 3b + a)^2 + (x + 3b + 2a)^2$$
$$= (x + b)^2 + (x + b + a)^2 + (x + 4b + 2a)^2$$

In a similar way, we derive

$$x^2 + (x + 5b + a)^2 + [x + 2(5b + a)]^2 + [x + a + 2(5b + a)]^2$$
$$+ [x + a + 3(5b + a)]^2 = (x + b)^2 + [(x + b) + (5b + a)]^2$$
$$+ [(x + b + a) + (5b + a)] + [(x + b + a) + 2(5b + a)]^2$$
$$+ [(x + b + a) + 3(5b + a)]^2$$

In further expansions using larger squares, successive coefficients of b in the second term are $7, 9, 11 \ldots 2y - 1$ (where y has the significance appearing in earlier identities).

Our final result is

$$\sum_{r=0}^{y-1}\{x + r[(2y - 1)b + a]\}^2 + \sum_{r=y-1}^{2y-3}\{(x + a) + r[(2y - 1)b + a]\}^2$$
$$= \sum_{r=0}^{y-2}\{(x + b) + r[(2y - 1)b + a]\}^2 + \sum_{r=y-2}^{2y-3}\{(x + b + a)$$
$$+ r[(2y - 1)b + a]\}^2$$

Other paired rows and columns in $(2n+1)^2$ magic squares will yield other identities.

It is probably not possible to deal with $(2n)^2$ magic squares in the same way.* However, the following, based on some 4×4 squares may be of interest:

$$x^2 + (x + 2b + a)^2 + (x + 2b + a + 2y)^2 + (x + 2b + 2a + 2y)^2$$
$$= (x + b)^2 + (x + b + a)^2 + (x + b + a + 2y)^2$$
$$+ (x + 3b + 2a + 2y)^2$$

If we take $a = b = y = 1$, we get

$$x^2 + (x+3)^2 + (x+5)^2 + (x+6)^2$$
$$= (x+1)^2 + (x+2)^2 + (x+4)^2 + (x+7)^2$$

This shows that any 8 successive numbers can be so arranged in two equal groups that the sums of the numbers in the two groups are equal and the sums of the squares of the numbers are equal.

[Other identities bring in higher and higher powers; e.g.

$$x^3 + (x+3)^3 + (x+5)^3 + (x+6)^3 + (x+9)^3 + (x+10)^3$$
$$+ (x+12)^3 + (x+15)^3 = (x+1)^3 + (x+2)^3 + (x+4)^3$$
$$+ (x+7)^3 + (x+8)^3 + (x+11)^3 + (x+13)^3 + (x+14)^3$$

This shows that any 16 successive numbers can be so arranged in two equal groups that the sums of the numbers, the sums of the squares of the numbers and the sums of the cubes of the numbers are equal. And so on ad infinitum with 32, 64 . . . 2^n successive numbers where the highest power involved is the $(n-1)$th].

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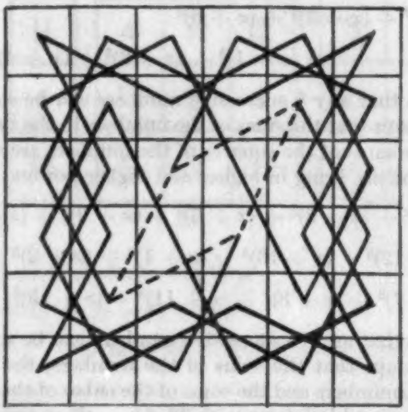
* Neither of the following qualifies, not being based on any magic squares I have seen or been able to construct:

$$\sum_{r=0}^{y}\{x + r(2y+5)\}^2 + \{x + 3 + y(2y+5)\}^2 + \sum_{r=y}^{2y-1}\{(x+6) + r(2y+5)\}^2$$
$$= \sum_{r=0}^{y-1}\{x + 2 + r(2y+5)\}^2 + \{x + 5 + (y-1)(2y+5)\}^2$$
$$+ \sum_{r=y-1}^{2y-1}\{(x+8) + r(2y+5)\}^2$$

$$x^2 + (x + 2y + 1)^2 + (x + 3y + 1)^2 \ldots + \ldots [x + (y+1)y + 1]^2$$
$$= (x + y - 1)^2 + (x + 2y - 1)^2 \ldots + \ldots [x + (y \cdot y) - 1]^2$$
$$+ [x + (y \cdot y) - 1 + (2y+1)]^2$$

2964. On Note 2761

The five solutions to the knight's tour problem given in Note 2761 are the only ones with a rotational four-fold symmetry. However, it is possible to construct semi-symmetrical solutions; instead of the rotational symmetry isomorphic to the permutation group (2341), one investigates the symmetry of double reflection in the two medians isomorphic to the Abelian group of order 4 generated by (12) (34) and (13) (24). Since this group has no operations of order 4, the resulting solutions must be in two closed mutually reflected circuits. These may, in two of the solutions, be linked together by a symmetric



treatment of the alternative connections shown dotted in the figure. The reader may be amused to find the other distinct solution.

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Marlow

2965. A Property of the Axioms of Projective Geometry

1. Projective geometry of two dimensions has the axioms of incidence:

(i) through any two points passes a unique straight line.
(ii) any two straight lines intersect in a unique point.

The elegance of these axioms leads us to try to find some analogous representation of them. Accordingly let us denote the concepts of 'point' and 'line' by the residue classes 1 and 2, respectively, modulo 3. Then if we think of the sign $\equiv$ as standing for 'determine,' the

equations

$$1 + 1 \equiv 2 \pmod 3,$$
$$2 + 2 \equiv 1 \pmod 3$$

may be read respectively:

two points determine a line,
two lines determine a point.

We see that our two equations give, in a convenient shorthand notation, the axioms of incidence of two-dimensional projective geometry.

It is natural to try to extend this notion. Let us now denote 'point,' 'line,' 'plane' by the residue classes 1, 2, 3, respectively, modulo 4. Then the equations

$$1 + 1 \equiv 2 \pmod 4,$$
$$1 + 2 \equiv 3 \pmod 4,$$
$$2 + 3 \equiv 1 \pmod 4,$$
$$3 + 3 \equiv 2 \pmod 4$$

may be read respectively:

two points determine a line,
a point and a line determine a plane,
a line and a plane determine a point,
two planes determine a line.

This gives us a shorthand representation of the axioms of incidence of projective geometry in three dimensions.

Thus if we now wish to venture into higher dimensions, we have the following empirical rule—for geometry of n dimensions, take for point, line, plane, . . . the residue classes 1, 2, 3, . . . modulo $(n + 1)$.

As an example, let us ask what is the intersection of two planes in *four* dimensions. Then the corresponding equation

$$3 + 3 \equiv 1 \pmod 5,$$

gives the answer as a point.

2. Duality.

We notice that the equations corresponding to the two and three dimensional cases, which have zero in the right-hand side, i.e.

$$1 + 2 \equiv 0 \pmod 3,$$
$$1 + 3 \equiv 0 \pmod 4,$$
$$2 + 2 \equiv 0 \pmod 4,$$

give us on the left-hand side the elements which are duals.

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2966. On spherical triangles

I am indebted to Dr. Campbell for the following proofs of the common spherical trigonometry formulae.

OA, OB and OC are unit vectors, and the triangle ABC is on the surface of the sphere, centre O. The axes Ox, Oy and Oz are chosen such that A lies on Oz and B is in the xz plane.

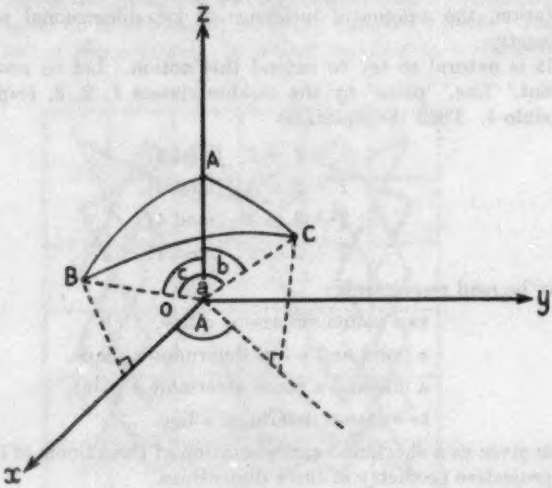

Fig. 1

Now

$$OB = \begin{bmatrix} \sin c \\ 0 \\ \cos c \end{bmatrix} \quad \text{and} \quad OC = \begin{bmatrix} \sin b \cos A \\ \sin b \sin A \\ \cos b \end{bmatrix}$$

Hence the scalar product

$$OB \,.\, OC = \cos c \cos b + \sin b \sin c \cos A.$$

But $OB \,.\, OC = \cos a$ since all are unit vectors.

Whence

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

We now rotate the system in Figure 1 about the y-axis until B lies on Ox and A lies in the xz plane. The new arrangement is shown in Figure 2.

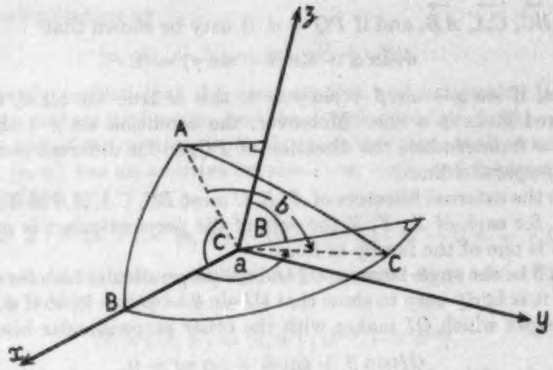

FIG. 2

In this new arrangement,

$$OC = \begin{bmatrix} \cos a \\ \sin a \sin B \\ \sin a \cos B \end{bmatrix}$$

We have rotated about the y-axis, and hence in the rotation from Figure I to Figure II the y-component of OC has remained unchanged.

Hence $\sin b \sin A = \sin a \sin B$

$$\text{or} \quad \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b}\left(= \frac{\sin C}{\sin c}\right).$$

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2967. A triangle property

Mrs. H. Spreadbury proposed the following problem. A point P moves on a line perpendicular to the line joining the circumcentre to the incentre of a triangle. Show that the algebraic sum of the perpendiculars from P to the sides of the triangle is constant.

It is more convenient to consider the problem the other way round; if the sum of the perpendiculars from P to the sides is constant, what is the locus of P?

First Solution: If two points P, Q are such that the sum of the perpendiculars from P to the sides is the same as the sum of the perpendiculars from Q to the sides then if $\overrightarrow{PQ}$ makes angles α, β, γ

with $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$, and if $PQ = d$, it may be shown that

$$d(\sin\alpha + \sin\beta + \sin\gamma) = 0.$$

Hence, if $\sin\alpha + \sin\beta + \sin\gamma = 0$, this is true for all d, so the required locus is a line. Moreover, the condition $\sin\alpha + \sin\beta + \sin\gamma = 0$ determines the direction of PQ, so for different sums the loci are parallel lines.

Let the external bisectors of A, B, C meet BC, CA, AB in X, Y, Z. Then, for each of X, Y, Z the sum of the perpendiculars is zero, so XYZ is one of the family of lines.

Let θ be the angle between OI and the perpendicular bisector of BC. Then it is fairly easy to show that $OI\sin\theta = \frac{1}{2}(c - b)$, so if ϕ, ψ are the angles which OI makes with the other perpendicular bisectors,

$$OI(\sin\theta + \sin\phi + \sin\psi) = 0.$$

Hence OI is a line perpendicular to the family of parallel lines.

Second Solution: If (x, y, z) are trilinear coordinates, so that $ax + by + cz = 2\Delta$, in the usual notation, the condition that the sum of the perpendiculars from P to the sides is constant is $x + y + z = k$. Hence the locus of P is the line whose equation is $2\Delta(x + y + z) = k(ax + by + cz)$. Different values of k give parallel lines, since the lines all pass through the point where $x + y + z = 0$ meets the line at infinity $ax + by + cz = 0$.

Now the circumcircle has equation $ayz + bzx + cxy = 0$, and the polar of I for the circumcircle has equation

$$(b + c)x + (c + a)y + (a + b)z = 0.$$

The polar is, of course, perpendicular to OI. This belongs to the system of parallel lines, as we see by taking $\dfrac{2\Delta}{k} = a + b + c$.

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2968. On the definition of dual numbers

Dual numbers are sometimes defined as expressions of the form $a + \varepsilon a'$, where a and a' are real numbers, and ε is an algebraic quantity such that $\varepsilon^2 = 0$. The following method of defining them is similar to one of the methods used to define complex numbers $x + iy$.

We define a dual number as an ordered pair (a, a') of real numbers. Two dual numbers (a, a') and (b, b') are said to be equal if, and only if, $a = b$ and $a' = b'$. Addition is defined by

$$(a, a') + (b, b') = (a + b, a' + b'),$$

and multiplication by

$$(a, a') \,.\, (b, b') = (ab, ab' + a'b). \tag{i}$$

It is easily verified that the commutative and associative laws of addition and multiplication, and the distributive law, hold.

The unit dual number is (1, 0), and the zero dual number is (0, 0). (a, a') has an additive inverse $(-a, -a')$, and subtraction is then defined by

$$(a, a') - (b, b') = (a, a') + (-b, -b') = (a - b, a' - b').$$

If $a \neq 0$, (a, a') has a multiplicative inverse $(a^{-1}, -a' \, a^{-2})$, and division is then defined by

$$(a, a')/(b, b') = (a, a') \,.\, (b^{-1}, -b' \, b^{-2}),$$

provided $b \neq 0$. Since division by $(0, b')$ is undefined for all b', not just $b' = 0$, the set of all dual numbers is not a field.

Now let ε denote the dual number (0, 1). Then by (i)

$$\varepsilon^2 = (0, 1) \,.\, (0, 1) = (0, 0).$$

Also
$$(a, a') = (a, 0) + (0, 1) \,.\, (a', 0). \tag{ii}$$

As a result of the above definitions, we may, for practical purposes, replace $(a, 0)$ by a for every real number a, although they are logically distinct. Hence from (ii) we can express a dual number in the form $a + \varepsilon a'$, where $\varepsilon^2 = 0$.

Another method of introducing dual numbers is to use the symbols $\begin{pmatrix} a & a' \\ 0 & a \end{pmatrix}$, with addition and multiplication defined as for matrices. The unit symbol, zero symbol, and ε are, respectively,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

A *dual vector* is formed from three dual numbers in the same way as an ordinary three-component vector is formed from three real numbers. Applications of dual vectors to geometry are given in references [1] and [2] below, and applications to mechanics in [3].

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REFERENCES

[1] J. G. Semple and L. Roth, "Introduction to Algebraic Geometry," Oxford, 1949, pp. 267–70.

[2] J. A. Todd, Math. Gazette 20 (1936), pp. 184–5.

[3] L. Brand, "Vector and Tensor Analysis," Wiley, New York, 1948, pp. 63–83 and 117–30.

2969. A note on the De-Moivre-Laplace Limit Theorem

The purpose of this note is to show the role which the well known formula

$$\sum_{v=0}^{k}\binom{n+1}{v}p^v q^{n-v+1} = (n-k+1)\binom{n+1}{k}\int_0^q t^{n-k}(1-t)^k\,dt, \quad (1)$$

can play in the proof of the de-Moivre-Laplace limit theorem in probability theory. The theorem can be formulated in the following manner:

$$\lim \sum_{v=0}^{k_n}\binom{n+1}{v}p^v q^{n-v+1} = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\alpha}\exp(-t^2/2)\,dt,$$

where $k_n = [np + \alpha\sqrt{npq}]$ and α is a real constant. Using (1) we have to find the limit of

$$I_n = \frac{(n+1)!}{k_n!(n-k_n)!}\int_0^q t^{n-k_n}(1-t)^{k_n}\,dt,$$

and by Stirling's formula we have

$$I_n = \frac{(n+1)\sqrt{n}(1+o(1))}{\sqrt{2\pi k_n(n-k_n)}}\int_0^q\left(\frac{t}{1-\frac{k_n}{n}}\right)^{n-k_n}\left(\frac{1-t}{\frac{k_n}{n}}\right)^{k_n} dt. \quad (2)$$

Since $k_n = np + \alpha\sqrt{npq} + \theta_n$, $0 \leq |\theta_n| < 1$ we find

$$\frac{k_n}{n} = p + \alpha\sqrt{\frac{pq}{n}} + \frac{\theta_n}{n} = p + \alpha_n\delta_n,\ n - k_n = q - \alpha_n\delta_n$$

where $\alpha_n \to \alpha$ and $\delta_n = (pq/n)^{\frac{1}{2}}$. With this notation and putting $t = q - \alpha_n\delta_n + x\delta_n$ (2) becomes

$$I_n = \gamma_n\int_{-\mu_n}^{\alpha_n}\varphi_n(x)\,dx$$

$$= \gamma_n\int_{-\mu_n}^{\alpha_n}\left(1+\frac{\delta_n x}{q-\alpha_n\delta_n}\right)^{n(q-\alpha_n\delta_n)}\left(1-\frac{\delta_n x}{p+\alpha_n\delta_n}\right)^{n(p+\alpha_n\delta_n)} dx \quad (3)$$

where

$$\mu_n = \sqrt{\frac{qn}{p}} - \alpha_n,\ \gamma_n = \frac{(n+1)\sqrt{pq}}{\sqrt{2\pi k_n(n-k_n)}} \to (2\pi)^{-\frac{1}{2}}.$$

Clearly $\varphi_n(x) \geqq 0$. Further

$$\log\varphi_n(x) = -\frac{x^2}{2}\cdot\frac{pq}{(p+\alpha_n\delta_n)(q-\alpha_n\delta_n)} + O\left(\frac{1}{\sqrt{n}}\right)$$

and

$$\lim_{n\to\infty} \varphi_n(x) = \exp(-x^2/2)$$

uniformly in every finite interval. Let z be chosen so that $z > (4pq)^{-1}$. Then it is easy to see that for sufficiently great n and for $-\mu_n \leq x \leq 0$ the function $\varphi_n(x) \exp(x^2/2z)$ has a positive derivative. But this leads to the inequality

$$\varphi_n(x) \leq \exp(-x^2/2z).$$

Now we can write

$$\begin{aligned}\Delta &= \int_{-\infty}^{\alpha} \exp(-x^2/2)\, dt - \int_{-\mu_n}^{\alpha_n} \varphi_n(x)\, dx \\ &= \int_{-\infty}^{-A} \exp(-x^2/2)\, dx + \int_{-A}^{\alpha_n} (\exp(-x^2/2) - \varphi_n(x))\, dx \\ &\quad + \int_{\alpha_n}^{\alpha} \exp(-x^2/2)\, dx - \int_{-\mu_n}^{-A} \varphi_n(x)\, dx\end{aligned}$$

Then we choose $A > 0$ so great that

$$\int_{-\infty}^{-A} \exp(-x^2/2)\, dx < \frac{\varepsilon}{4} \text{ and } \int_{-\infty}^{-A} \exp(-x^2/2z)\, dx < \frac{\varepsilon}{4}$$

and further n so great that

$$\int_{-\mu_n}^{-A} \varphi_n(x)\, dx \leq \int_{-\infty}^{-A} \exp(-x^2/2z)\, dt < \frac{\varepsilon}{4}, \quad \int_{\alpha_n}^{\alpha} \exp(-x^2/2)\, dx < \frac{\varepsilon}{4},$$

$$\int_{-A}^{\alpha_n} |\exp(-x^2/2) - \varphi_n(x)|\, dx < \frac{\varepsilon}{4}.$$

This gives $|\Delta| < \varepsilon$ as required.

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2970. A fundamental inequality for integrals

We give a very simple proof that the Riemann integral of a positive function is positive.

Let $f(x) > 0$ have an upper Riemann integral over $[a, b]$. If for every $\varepsilon > 0$ and for every interval i contained in $[a, b]$ there is an interval j contained in i such that

$$\overline{\mathrm{bd}}_{x\in j} f(x) < \varepsilon$$

then we may determine a sequence of intervals $i_{n+1} \subset i_n \subset [a, b]$ such that

$$\overline{\underset{x \in i_n}{bd}} < \frac{1}{n}$$

The nest of intervals i_n contains a unique point ξ and therefore $0 < f(\xi) < 1/n$, for every n, and this contradiction proves that there is an $\varepsilon_0 > 0$ and an interval $i_0 = [c, d] \subset (a, b)$ such that for every interval $j \subset i_0$

$$\overline{\underset{x \in j}{bd}}\, f(x) \geq \varepsilon_0$$

and therefore

$$\overline{\int_a^b} f(t)dt \geq \overline{\int_c^d} f(t)dt \geq \varepsilon_0(d - c) > 0$$

J. St-C. L. Sinnadurai

2971. The equation of a (1, 1) correspondence

In the Gazette, Vol. XXIII, p. 58, (1939), J. A. Todd gave a proof of the theorem than an algebraic relation giving a (1, 1) correspondence between two variables x and y, in which infinite values are admitted, can be reduced to the form $axy + bx + cy + d = 0$. The following proof appears to be more direct.

Let the relation be $f(x, y) = 0$. We may suppose it to have been put into its polynomial form. Accordingly, we write

$$f(x, y) \equiv \sum_{r=0}^{n} \binom{n}{r} a_r(x) y^{n-r} \tag{1}$$

where a_r are polynomials in x.

When $y = \infty$, (1) takes the form $a_0(x) = 0$, and since this gives a unique value of x, say x_1, $a_0(x) = u^n(x - x_1)^P$, where P is a positive integer and u is a non-zero constant. Similarly, $a_n(x) = v^n(x - x_2)^Q$, where Q is a positive integer, v is a non-zero constant and x_2 is the value of x when $y = 0$.

Let X, Y be any pair of finite corresponding values of x, y. Then, since X differs from x_1, $a_0(X) \neq 0$, and, since $f(X, y) = 0$ gives $y = Y$ uniquely,

$$f(X, y) = a_0(X) \,.\, (y - Y)^n \tag{2}$$

for all values of y. On equating coefficients of powers of y between (2) and the result of setting $x = X$ in (1), we have

$$a_r(X) = (-Y)^r a_0(X), \; r = 0 \text{ to } n,$$

and therefore the equations

$$\frac{a_0(x)}{a_1(x)} = \frac{a_1(x)}{a_2(x)} = \ldots = \frac{a_{n-1}(x)}{a_n(x)} = \left(\frac{a_0(x)}{a_n(x)}\right)^{1/n} \tag{3}$$

are satisfied by every $x \neq x_1, x_2$ and are therefore identities in x. Hence $\left(\dfrac{a_0(x)}{a_n(x)}\right)^{1/n}$ is a rational function and so $P = np$, $Q = nq$, where p, q are positive integers. From (3), $a_r(x) = u^{n-r}v^r(x - x_1)^{(n-r)p}(x - x_2)^{rq}$, and it follows that $f(x, y) \equiv \{u(x - x_1)^p y + v(x - x_2)^q\}^n$, so that the relation reduces to $u(x - x_1)^p y + v(x - x_2)^q = 0$, i.e. to a form linear in y.

Let this reduced relation be expressed in the form $\sum_{r=0}^{m} \binom{m}{r} b_r(y)x^{m-r} = 0$ where m is the larger of p, q, and where b_r are at most linear in y. The previous argument applied to the new form shows that $b_0(y)$ and $b_m(y)$ are not y-free and that $\left(\dfrac{b_0(y)}{b_m(y)}\right)^{1/m}$ is rational, so that $m = 1$ and the relation becomes

$$u(x - x_1)y + v(x - x_2) = 0,$$

which is of the prescribed form.

The argument breaks down if for instance $x_1 = \infty$. In this case the theorem may be proved by first making the substitution $x = \dfrac{1}{\xi} + k$, where $k \neq x_2$, so that $\xi_1 = 0$ and $\xi_2 \neq \infty$. Other cases of failure may be handled similarly.

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2972. Rational approximations to $(1 + x)^{\pm\frac{1}{2}}$ by iteration

If a is an approximation to $\sqrt{N}$, it is well known that

$$\tfrac{1}{2}(a + N/a) \tag{1}$$

is generally a better; see, for instance, Bowman's Elementary Algebra, part II, p. 83. A proof consists in showing that the error is equal to

$$(a - \sqrt{N})^2/2a, \tag{2}$$

which is smaller than $a - \sqrt{N}$ in all cases of interest. Successive application of this formula gives rapid convergence to $\sqrt{N}$, since the error after n iterations is of the order of the 2^nth power of the initial error.

If x is small, 1 is an approximation to $(1 + x)^{1/2}$. The above formula then gives the following successive approximations:

$$1 + \tfrac{1}{2}x, \quad \frac{1 + x + \frac{1}{8}x^2}{1 + \frac{1}{2}x}, \quad \frac{1 + 2x + \frac{5}{4}x^2 + \frac{1}{4}x^3 + \frac{1}{128}x^4}{(1 + \frac{1}{2}x)(1 + x + \frac{1}{8}x^2)}, \tag{3}$$

with errors of the order of $x^2, x^4, x^8, \ldots$ respectively. On replacing x by $2x$, the second of these becomes the approximation to $(1 + 2x)^{1/2}$

discussed by Mr. Homer in Note 2854, to which I am indebted for the stimulus to look further into such approximations.

The errors in these as approximations to $(1+x)^{1/2}$ can be assessed algebraically, without the mean value theorem Mr. Homer used, by appeal to (2). If $N = 1 + x$, $a_0 = 1$, and $a_1, a_2, a_3, \ldots$ are the successive functions (3), the errors may be found using (2); they are

$$
\left.
\begin{aligned}
a_1 - \sqrt{N} &= (a_0 - \sqrt{N})^2/2a_0 = \frac{\frac{1}{2}x^2}{\{1+\sqrt{(1+x)}\}^2} \\
a_2 - \sqrt{N} &= (a_0 - \sqrt{N})^4/2^3a_0{}^2a_1 = \frac{\frac{1}{8}x^4}{\{1+\sqrt{(1+x)}\}^4}\cdot\frac{1}{1+\frac{1}{2}x} \\
a_3 - \sqrt{N} &= (a_0 - \sqrt{N})^8/2^7a_0{}^4a_1{}^2a_2 = \frac{\frac{1}{128}x^8}{\{1+\sqrt{(1+x)}\}^8} \\
&\qquad \times \frac{1}{(1+\frac{1}{2}x)(1+x+\frac{1}{8}x^2)}
\end{aligned}
\right\} \tag{4}
$$

and so on. These formulae give the errors for all relevant values of x, whether small or not, since they are obtained by algebra only.

The errors are clearly always positive; and we easily find that

$$
a_1 - \sqrt{N} \le \tfrac{1}{2}x^2, \quad a_2 - \sqrt{N} \le \tfrac{1}{4}x^4, \quad a_3 - \sqrt{N} \le \tfrac{1}{8}x^8, \tag{5}
$$

since $x \ge -1$ for the existence of $\sqrt{(1+x)}$ as a real number. These inequalities can be improved as x moves away from -1; for instance, if $x > 0$,

$$
a_1 - \sqrt{N} < \frac{x^2}{2^3}, \quad a_2 - \sqrt{N} < \frac{x^4}{2^7}, \quad a_3 - \sqrt{N} < \frac{x^8}{2^{15}}. \tag{6}
$$

STRUCTURE OF THE APPROXIMATIONS. Because of the simplicity of the error formulae it seems worth while to formulate the nth approximation. From (3) it looks probable that

$$
a_n = p_{n-1}/p_0p_1p_2\ldots p_{n-2},
$$

where p_r is a polynomial in x of degree 2^r. We establish this as follows.

LEMMA 1. *If p_r are determined successively by*

$$
p_0 = 1 + \tfrac{1}{2}x, \quad p_n = p_{n-1}^2 - 2(\tfrac{1}{4}x)^{2^n}, \tag{7}
$$

then p_n is a polynomial of degree 2^n (no less), and

$$
(1+x)p_0{}^2p_1{}^2p_2{}^2\ldots p_{n-2}^2 = p_n - 2(\tfrac{1}{4}x)^{2^n}. \tag{8}
$$

The case $n = 2$ of (8) is verified by showing that both sides reduce

to $(1 + x)(1 + \frac{1}{2}x)^2$. The cases $n > 2$ are then proved by induction: supposing (8) holds as it stands, we have, by (8) and (7),

$$(1 + x)p_0^2p_1^2 \cdots p_{n-2}^2p_{n-1}^2 = \{p_n - 2(\tfrac{1}{4}x)^{2^n}\}\{p_n + 2(\tfrac{1}{4}x)^{2^n}\}$$
$$= p_n^2 - 4(\tfrac{1}{4}x)^{2^{n+1}} = p_{n+1} - 2(\tfrac{1}{4}x)^{2^{n+1}}.$$

That p_n is a polynomial of degree at most 2^n then follows from (8) by induction. The degree of the left side is at most

$$1 + 2(1 + 2 + 2^2 + \dots + 2^{n-2}) = 2^n - 1;$$

so that the right side has this degree at most, and the leading term in p_n is, in fact, $2(\frac{1}{4}x)^{2^n}$. So the proof of Lem. 1 is complete.

THEOREM 1. *If p_r are the polynomials of Lem.* 1, *the nth approximation to* $\sqrt{(1 + x)}$ *in the sequence* (3) *is*

$$a_n = p_{n-1}/p_0p_1p_2 \cdots p_{n-2}, \tag{9}$$

and the error in it is

$$a_n - \sqrt{(1 + x)} = \frac{2}{p_0p_1p_2 \cdots p_{n-2}}\left(\frac{\frac{1}{2}x}{1 + \sqrt{(1 + x)}}\right)^{2^n}. \tag{10}$$

These formulae are easily verified in the case $n = 2$ by use of (3), (4), (7). Induction proofs are completed as follows, supposing (9) and (10) hold for a particular value of n:

$$a_{n+1} = \tfrac{1}{2}\left(a_n + \frac{1 + x}{a_n}\right) = \frac{p_{n-1}^2 + (1 + x)p_0^2p_1^2 \cdots p_{n-2}^2}{2p_0p_1 \cdots p_{n-2}p_{n-1}}$$
$$= \frac{p_n + 2(\frac{1}{4}x)^{2^n} + p_n - 2(\frac{1}{4}x)^{2^n}}{2p_0p_1 \cdots p_{n-2}p_{n-1}} = \frac{p_n}{p_0p_1 \cdots p_{n-1}},$$

in which the central step uses Lem. 1, both (7) and (8). Again, using (2),

$$a_{n+1} - \sqrt{(1 + x)} = \{a_n - \sqrt{(1 + x)}\}^2/2a_n$$
$$= \frac{p_0p_1 \cdots p_{n-2}}{2p_{n-1}}\left(\frac{2}{p_0p_1 \cdots p_{n-2}}\right)^2\left(\frac{\frac{1}{2}x}{1 + \sqrt{(1 + x)}}\right)^{2^{n+1}}$$

which gives the equation got by formally replacing n by $n + 1$ in (10). So Thm. 1 is proved.

ESTIMATES OF THE ERRORS. We now obtain inequalities which generalize (5) and (6), bounding the error in the nth approximation a_n. At the same time we find that a_n decreases monotonically for each fixed value of x. As before we suppose that $x \geq -1$, since otherwise $\sqrt{(1 + x)}$ is not real.

LEMMA 2. *The polynomials $p_r(x)$ of Lem. 1 are positive and increasing throughout $x \geq -1$.*

This is evident for $r = 0$; assume it true for $r = 1, 2, \ldots, n-1$ also. By Lem. 1,

$$p_n = (1+x)p_0^2 p_1^2 p_2^2 \ldots p_{n-2}^2 + 2(\tfrac{1}{4}x)^{2^n}, = p_{n-1}^2 - 2(\tfrac{1}{4}x)^{2^n}.$$

The former shows that p_n is positive in $x \geq -1$ and increasing in $x \geq 0$; while the latter shows that p_n is increasing in $-1 \leq x \leq 0$. This proves Lem. 2.

THEOREM 2. *The approximations a_n of Thm. 1 satisfy, for all $x \geq -1$,*

$$0 \leq a_n - \sqrt{(1+x)} \leq \frac{x^{2^n}}{2^n}, \quad 1 - \tfrac{1}{2}x^{2^n} \leq \frac{a_{n+1}}{a_n} \leq 1; \qquad (11)$$

further, for $x \geq 0$,

$$a_n - \sqrt{(1+x)} \leq 2(\tfrac{1}{4}x)^{2^n}, \quad 1 - 2(\tfrac{1}{4}x)^{2^n} \leq a_{n+1}/a_n. \qquad (12)$$

Let $x = -1$. If $p_{n-1} = 2^{1-2^n}$, which is true for $n = 1$ by (7), then also by (7),

$$p_n = 2^{2(1-2^n)} - 2(-\tfrac{1}{4})^{2^n} = 4^{-2^n}(4-2) = 2^{1-2^{n+1}};$$

this establishes this value of p_n at $x = -1$ for all n.

Now suppose $x \geq -1$. By Lem. 2, $p_n(x) \geq p_n(-1) = 2^{1-2^{n+1}}$. So (10) gives

$$\begin{aligned} 0 \leq a_n - \sqrt{(1+x)} &\leq 2 \,.\, 2^{2-1} \,.\, 2^{4-1} \,.\, 2^{8-1} \ldots 2^{2^{n-1}-1}(\tfrac{1}{2}x)^{2^n} \\ &= 2^{1+2+4+8+\ldots+2^{n-1}-(n-1)-2^n}x^{2^n} = 2^{-n}x^{2^n}. \end{aligned}$$

And (9) gives, using (7) also,

$$\frac{a_{n+1}}{a_n} = \frac{p_n}{p_{n-1}^2} = 1 - \frac{2(\frac{1}{4}x)^{2^n}}{p_{n-1}^2} \geq 1 - \frac{2(\frac{1}{4}x)^{2^n}}{(2^{1-2^n})^2} = 1 - \tfrac{1}{2}x^{2^n},$$

which proves (11).

Again, let $x = 0$. If $p_{n-1} = 1$, which is true for $n = 1$ by (7), we also have that $p_n = 1$; so this holds for all n.

For $x \geq 0$ we have, by Lem. 2, $p_n(x) \geq p_n(0) = 1$. Using (10), (9) and (7) as above, but more simply, we obtain (12).

PARTIAL FRACTIONS FOR THE RATIONAL FUNCTIONS $1/a_n$. We obtain these partial fractions with the aim of relating $1/a_n$, which approximates $(1+x)^{-1/2}$, with the binomial expansion of this function. This turns out to be simple and elegant. The whole undertaking provides incidentally an algebraic proof of the binomial theorem for $(1+x)^{-1/2}$.

LEMMA 3. *The numbers τ_{nr} defined in Thm. 3 (below) are 2^n different real numbers, strictly between 0 and 4.*

This is true for $n = 1$, since τ_{11} and τ_{12} are $2 \pm \sqrt{2}$.

If it is true for a particular value of n, $\sqrt{\tau_{nr}}$ are 2^n different real numbers strictly between 0 and 2. Consequently $2+\sqrt{\tau_{nr}}$ are 2^n different real numbers between 2 and 4, and $2-\sqrt{\tau_{nr}}$ are 2^n different real numbers between 0 and 2. These 2^{n+1} different real numbers between 0 and 4 are the numbers $\tau_{n+1,r}$, and this proves the result.

THEOREM. 3. *If $\tau_{n1}, \tau_{n2}, \ldots, \tau_{n2^n}$ are the numbers*

$$2 \pm \sqrt{(2 \pm \sqrt{(2 \pm \ldots \sqrt{2} \ldots))}$$

in which there are n square root signs (even if $n = 0$), then for $x \geq -1$

$$\frac{1}{a_{n+1}(x)} = \frac{1}{2^n}\sum_{r=1}^{2^n} \frac{1}{1+\frac{1}{4}\tau_{nr}x}. \tag{13}$$

And for all x

$$p_n(x) = \prod_{r=1}^{2^n}(1+\tfrac{1}{4}\tau_{nr}x). \tag{14}$$

These statements are evident from (3) and (7) if $n = 0$; so we may suppose n is a positive integer.

Splitting the summation into two blocks of 2^{n-s-1} terms each

$$\begin{aligned}
&\sum_{r=1}^{2^{n-s}} \frac{1}{p_s + (2-\tau_{n-s,r})(\frac{1}{4}x)^{2^s}} \\
&= \sum_{r=1}^{2^{n-s-1}} \left(\frac{1}{p_s - \sqrt{\tau_{n-s-1,r}}(\frac{1}{4}x)^{2^s}} + \frac{1}{p_s + \sqrt{\tau_{n-s-1,r}}(\frac{1}{4}x)^{2^s}}\right) \\
&= \sum_{r=1}^{2^{n-s-1}} \frac{2p_s}{p_s^2 - \tau_{n-s-1,r}(\frac{1}{4}x)^{2^{s+1}}} \\
&= 2p_s \sum_{r=1}^{2^{n-s-1}} \frac{1}{p_{s+1} + (2-\tau_{n-s-1,r})(\frac{1}{4}x)^{2^{s+1}}}
\end{aligned}$$

using (7). No denominator vanishes if none of those in the first summation vanish.

Applying this result for $s = 0, 1, \ldots, n-1$ in succession,

$$\begin{aligned}
\sum_{r=1}^{2^n} \frac{1}{1+\frac{1}{4}\tau_{nr}x} &= \sum_{r=1}^{2^n} \frac{1}{p_0 - (2-\tau_{nr})\frac{1}{4}x} \\
&= \sum_{r=1}^{2^n} \frac{1}{p_0 + (2-\tau_{nr})\frac{1}{4}x} \\
&= 2^n p_0 p_1 p_2 \cdots p_{n-1} \sum_{r=1}^{2^0} \frac{1}{p_n + (2-\tau_{0r})(\frac{1}{4}x)^{2^n}} \\
&= 2^n \frac{p_0 p_1 p_2 \cdots p_{n-1}}{p_n}
\end{aligned} \tag{15}$$

The only condition needed is that none of the denominators on the left vanishes, that is, $x \neq -4/\tau_{nr}$. This is certainly fulfilled if $x \geq -1$, since, by Lem. 3, $-4/\tau_{nr} < -1$. Using (9), (15) gives (13).

The numerator on the right of (15) is a polynomial; consequently p_n vanishes at all the poles of the left side, at least. These poles are, by Lem. 3, 2^n different numbers $-4/\tau_{nr}$. Since p_n is of degree 2^n it has these zeros and no others; and these are all simple. Also $p_n(0) = 1$, as observed in proving Thm. 2; whence follows (14). The restriction $x \geq -1$ is of course unnecessary for it.

COROLLARY. *The polynomial $p_n(x)$ has 2^n different real roots, all satisfying $x < -1$ and therefore outside the range of approximation.*

All coefficients in $p_n(x)$ are positive; and this can still be said when $p_n(x)$ is re-written as a polynomial in $x + 1$.

A more direct proof of (14) might naturally be expected. It is purely algebraic, not involving the slightly analytic idea of "pole". It shows the origin of the numbers τ_{nr}, and makes Thm. 3 less unexpected.

To simplify the difference equation (7), write

$$q_n(y) = y^{2^n} p_n(-4/y). \tag{16}$$

This reduces (7) to

$$q_0 = y - 2, \quad q_n = q_{n-1}^2 - 2;$$

whence

$$q_n(y) = [\,.\,.\,\{(y-2)^2 - 2\}^2 - 2\,.\,.\,]^2 - 2, \tag{17}$$

with n squarings. Thus $q_n(y)$ is a polynomial of degree 2^n with leading coefficient 1; and it vanishes at $y = \tau_{nr}$, 2^n different numbers by Lem. 3. So

$$q_n(y) = \prod_{r=1}^{2^n} (y - \tau_{nr}). \tag{18}$$

By putting $y = -4/x$ and applying (16), (18) is translated into (14).

A BINOMIAL IDENTITY. This item appears to be quite detached from the preceding work, but its connection will be seen presently.

LEMMA 4. *If* $$W_n = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} = (-1)^n\binom{-\frac{1}{2}}{n} \tag{19}$$

(W for Wallis) and ${}^nC_r$ are the simple binomial coefficients, then

$$1 + {}^nC_2 W_1 + {}^nC_4 W_2 + \ldots = 2^n W_n. \tag{20}$$

By the simple binomial theorem,

$$1 + {}^nC_1\{\tfrac{1}{2}(x^2 + x^{-2})\} + {}^nC_2\{\tfrac{1}{2}(x^2 + x^{-2})\}^2 + {}^nC_3\{\tfrac{1}{2}(x^2 + x^{-2})\}^3 + \ldots$$
$$= \{1 + \tfrac{1}{2}(x^2 + x^{-2})\}^n$$
$$= 2^{-n}\{x^2 + 2 + x^{-2}\}^n = 2^{-n}(x + x^{-1})^{2n}.$$

The constant term on the right is

$$2^{-n} \,.\, {}^{2n}C_n = 2^{-n}\frac{(2n)!}{n!n!} = \frac{1\,.\,3\,.\,5\,.\,.\,.\,(2n-1)}{n!} = 2^n W_n,$$

and this is equal to the constant term on the left, which is

$$1 + \sum_{r=1}^{[\frac{1}{2}n]} {}^nC_{2r} \,.\, 2^{-2r} \,.\, {}^{2r}C_r = 1 + \sum_{r=1}^{[\frac{1}{2}n]} {}^nC_{2r} W_r.$$

An alternative proof, not algebraic, is given by evaluating the integral

$$\int_0^\pi (1 + \cos\theta)^n \, d\theta,$$

firstly using the binomial expansion of the integrand and secondly by rewriting the integrand as $2^n \cos^{2n}\frac{1}{2}\theta$, in each case using a well-known reduction formula.

Prof. Wilansky's "Genesis for Binomial Identities" (Math. Gaz. vol. XLIII no. 345, Oct. 1959, p. 176–7) does not seem to contain this identity, because the last term on the left involves $W_{[\frac{1}{2}n]}$ rather than W_n. By taking $g(t) = 1/\sqrt{(1-t^2)}$ in his Thm. A we obtain

$$1 - {}^nC_1W_1 + {}^nC_2W_2 - {}^nC_3W_3 + \ldots = W_n.$$

It seems unlikely that this is closely related to (20), although it can also be proved by the method of Lem. 4, using the identity

$$1 - {}^nC_1\{\tfrac{1}{2}(x + x^{-1})\}^2 + {}^nC_2\{\tfrac{1}{2}(x + x^{-1})\}^4 - \ldots = \{1 - \tfrac{1}{4}(x + x^{-1})^2\}^n.$$

AN INEQUALITY WITH MANY CASES OF EQUALITY. This item also stands on its own, except in relating to the roots of (17) and in using Lem. 4.

LEMMA 5. *If τ_{nr} are the numbers $2 \pm \sqrt{(2 \pm \sqrt{(2 \pm \ldots \sqrt{2}\,.\,.))}}$ in which there are n square root signs, and W_k are the numbers defined in* (19), *then*

$$\sum_{r=1}^{2^n} \tau_{nr}^k \leq 2^{n+2k} W_k. \tag{21}$$

There is EQUALITY *for every positive integer $k < 2^{n+1}$.*

In the case $n = 0$, the inequality is proved by

$$2^{2k}W_k = 2^k \cdot \frac{1}{1} \cdot \frac{3}{2} \cdot \frac{5}{3} \cdots \frac{2k-1}{k} \geq 2^k = \tau_{01}^k = \sum_{r=1}^{2^0} \tau_{0r}^k,$$

and there is clearly equality if $k = 1$.

Suppose the lemma holds for a particular value of n, possibly 0

Denoting the summation on the left of (21) by σ_{nk},

$$\begin{aligned}
\sigma_{n+1,k} &= \sum_{r=1}^{2^n}(2+\sqrt{\tau_{nr}})^k + \sum_{r=1}^{2^n}(2-\sqrt{\tau_{nr}})^k \\
&= 2\sum_{r=1}^{2^n}(2^k + {}^kC_2 2^{k-2}\tau_{nr} + {}^kC_4 2^{k-4}\tau^2_{nr} + \ldots) \\
&= 2(2^{n+k} + {}^kC_2 2^{k-2}\sigma_{n1} + {}^kC_4 2^{k-4}\sigma_{n2} + \ldots) \qquad (22) \\
&\leq 2(2^{n+k} + {}^kC_2 2^{k-2} 2^{n+2} W_1 + {}^kC_4 2^{k-4} . 2^{n+4} W_2 + \ldots) \\
&= 2^{1+n+k}(1 + {}^kC_2 W_1 + {}^kC_4 W_2 + \ldots) \\
&= 2^{1+n+k} . 2^k W_k
\end{aligned}$$

by Lem. 4; this proves (21) with n replaced by $n+1$.

Suppose $k < 2^{n+2}$. The terms in (22) after the first are ${}^kC_{2h}2^{k-2h}\sigma_{nh}$, with positive integers $h \leq \frac{1}{2}k < 2^{n+1}$. So the cases of (21) used in proceeding from (22) to the next line are all equalities. Thus $\sigma_{n+1,k} = 2^{1+n+2k}W_k$ is established, for these values of k. This completely proves the lemma.

THE BINOMIAL EXPANSION OF $(1+x)^{-1/2}$. We can now show that Thm. 3 leads to the partial sums of the binomial series, or rather those having 2^n terms, as polynomial approximations to $(1+x)^{-1/2}$; and we obtain estimates of the error which are closely comparable with the usual ones obtained assuming convergence to $(1+x)^{-1/2}$ of the binomial series.

THEOREM 4. *If a_n is the nth approximation to $(1+x)^{1/2}$ defined in (7) and (9), s_m is the sum of the first m terms of the binomial series for $(1+x)^{-1/2}$, and $m = 2^n$, then*

$$0 \leq (1+x)^{-1/2} - \frac{1}{a_n} \leq \frac{1}{m}\cdot\frac{x^m}{1+x}, \quad \text{or} \quad \frac{1}{2^{2m-1}}\,x^m, \qquad (23)$$

$$0 \leq \frac{1}{a_n} - s_m \leq \frac{1}{1-(1/2m)}\binom{-\frac{1}{2}}{m}\frac{x^m}{1+x}, \quad \text{or} \quad \binom{-\frac{1}{2}}{m}x^m. \qquad (24)$$

These inequalities apply for all $x > -1$; except that the alternative right hand sides apply only for all $x \geq 0$.

Thm. 2 shows that $a_n \geq \sqrt{(1+x)} \geq 0$, so that

$$0 \leq \frac{1}{\sqrt{(1+x)}} - \frac{1}{a_n} = \frac{a_n - \sqrt{(1+x)}}{a_n\sqrt{(1+x)}} \leq \frac{a_n - \sqrt{(1+x)}}{1+x},$$

from which (23) follows by further application of Thm. 2.

Thm. 3 gives the following expression for $1/a_{n+1}$

$$\frac{1}{m}\sum_{r=1}^{m}\left\{1-\tfrac{1}{4}\tau_{nr}x+(\tfrac{1}{4}\tau_{nr}x)^2-\ldots-(\tfrac{1}{4}\tau_{nr}x)^{2m-1}+\frac{(\tfrac{1}{4}\tau_{nr}x)^{2m}}{1+\tfrac{1}{4}\tau_{nr}x}\right\}$$

$$=1-W_1x+W_2x^2-\ldots-W_{2m-1}x^{2m-1}+\frac{x^{2m}}{2^{n+4m}}\sum_{r=1}^{m}\frac{\tau_{nr}{}^{2m}}{1+\tfrac{1}{4}\tau_{nr}x}$$

using Lem. 5 in all the cases of equality.

If $x \geq 0$ this gives, using Lem. 3 and 5,

$$0 \leq \frac{1}{a_{n+1}}-s_{2m} \leq \frac{x^{2m}}{2^{n+4m}}\sum_{r=1}^{m}\tau_{nr}^{2m} \leq x^{2m}W_{2m}=\binom{-\frac{1}{2}}{2m}x^{2m},$$

even in the case $n = 0$. Replacing n by $n - 1$, and consequently m by $\frac{1}{2}m$, we obtain the second version of (24) for all positive integers n.

If $x > -1$, we use Lem. 3 and 5 a little differently:

$$\frac{1}{a_{n+1}}-s_{2m}=\frac{x^{2m}}{2^{n+4m}}\sum_{r=1}^{m}\frac{\tau_{nr}^{2m}}{(1-\frac{1}{4}\tau_{nr})+\frac{1}{4}\tau_{nr}(1+x)} \geq 0$$

$$\leq \frac{x^{2m}}{2^{n+4m}}\sum_{r=1}^{m}\frac{4\tau_{nr}^{2m-1}}{1+x}$$

$$=\frac{x^{2m}}{1+x}W_{2m-1}=\frac{x^{2m}}{1+x}\cdot\frac{4m}{4m-1}\cdot W_{2m},$$

which as before gives the first version of (24) for all positive integers n.

COROLLARY. *The remainder after* 2^n *terms of the binomial series for* $(1 + x)^{-1/2}$ *is positive and less than*

$$\left\{\frac{1}{1-(1/2m)}\binom{-\frac{1}{2}}{m}+\frac{1}{m}\right\}\frac{x^m}{1+x},$$

where $m = 2^n$, *for all* $x > -1$.

Also $1/a_n$ *is a better approximation to* $(1 + x)^{-1/2}$ *than the sum of* 2^n *terms of the binomial expansion, although the error in both is* $O(x^{2^n})$.

University of Melbourne E. R. LOVE

2973. A note on the game of Nim

In Note 2334, Professor Alan S. C. Ross points out that the name of the "Chinese game of Nim"[1] is unlikely to be Chinese. Nor, from what he there writes, does there appear to be any real evidence to connect the game with China; and the literature of Games nowhere appears to make any other suggestions about its origin.

[1] So called by G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (3rd ed.), p. 116 ff.

A possible relationship may be traced for Nim with the board-game of the type known as "mancala."[2] Under various names, and with many small variations in the mode of play, this game is well known in the Middle East, the Far East and Africa, but not in Europe or America (except among certain American negroes, who have brought the game with them from Africa; "wari" is a name for mancala which is used in the West Indies and West Africa). Play takes place on two, three or four rows of holes or cups, each of which contains a (generally equal) number of stones, beans etc. Each player takes in his turn the entire contents of one hole. He puts one bean into the first hole adjacent, one into the next hole, and so on, round the board, until the beans picked up are exhausted. The purpose of the turn is to contrive to place the final bean so that this hole contains a given number of beans (often, four), which are then captured by this player. The player who finishes with the greater number of beans is the winner.

The game of Nim is, of course, not a mancala game, in that it does not employ a board, cups or holes; and the preoccupation with remainders is of a different type. However, if we assume it to be a European/American derivative of mancala, it is easy to understand that the rather cumbersome board should have been dispensed with, and the manner of play modified accordingly. It is interesting that Murray[3] suggests Egypt or Arabia as the original home of mancala; this would account for a supposed eastern origin for Nim; and it is not impossible that the name of Nim is a corruption of the first element of Arabic *minqala* 'mancala', from the verb *naqala* 'to move'.

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[2] See H. J. R. Murray, *A History of Board-games*, p. 158 ff.
[3] *op. cit.*, p. 158.

ERIC HAROLD NEVILLE

E. H. Neville died on August 23rd in his 73rd year after a short illness. Professor Neville joined the Mathematical Association in 1919, was President in 1934, Hon. Librarian from 1923 to 1954 and Editor of the Mathematical Gazette in 1930. An appreciation of his great services to the Association will appear in a later issue of this *Gazette*.

CLASS ROOM NOTES

72. Some expansions.

1. **Expansion of $\cos n\theta$ in powers of $\cos\theta$.**

Write C for $2\cos\theta$, C_n for $2\cos n\theta$ (notice factor 2) then $C_n = C \cdot C_{n-1} - C_{n-2}$ gives

$$C_2 = C^2 - 2,\quad C_3 = C^3 - 3C,\quad C_4 = C^4 - 4C^2 + 2$$

Ignore the signs of the coefficients and write them in a table:

1	2			
1	3			
1	4	2		
1	5	5		
1	6	9	2	
1	7	14	7	
1	8	20	16	2
1	9	27	30	9

We can easily prove the rule for construction

$$\begin{array}{c|c} a & \\ \hline & b \\ \hline & c \end{array} \qquad c = a + b$$

The last line gives e.g.

$$C_9 = C^9 - 9C^7 + 27C^5 - 30C^3 + 9C.$$

2. **Write $S_n = \sin n\theta/\sin\theta$, then**

$$S_n = CS_{n-1} - S_{n-2},\quad S_2 = C,\quad S_3 = C^2 - 1$$

We find a table with the same rule for construction but with a different start:

1					
1	1				
1	2				
1	3	1			
1	4	3			
1	5	6	1		
1	6	10	4		
1	7	15	10	1	
1	8	21	20	5	
1	9	28	35	15	1

$\sin 5\theta/\sin\theta = C^4 - 3C^2 + 1$ from fourth line

$$\sin 6\theta/\sin\theta = C^5 - 4C^3 + 3C$$

$$\sin 11\theta/\sin\theta = C^{10} - 9C^8 + 28C^6 - 35C^4 + 15C^2 - 1$$

The lines inclined at 45° to the downwards vertical give Pascal's triangle. This is confirmed by the method of formation, and can be used to yield the general formula.

3. **We often wish to arrange $(x^{2n+1} - 1)/(x - 1)$ in powers of $x + x^{-1}$. A direct attack even for small n leads to tedious work.**

$$\text{If}\quad \frac{x^r - 1}{x - 1} = 0 \quad\text{then}\quad \frac{x^{r/2} - x^{-r/2}}{x^{1/2} - x^{-1/2}} = 0$$

If $\cos\theta + i\sin\theta$ is a solution of $x^r - 1 = 0$, the last equation gives

$$\sin\frac{r\theta}{2}\Big/\sin\frac{\theta}{2} = 0.$$

Take $r = 2n + 1$ then $\sin(n + 1)\theta + \sin n\theta = 0$, and the last table gives the coefficients.

For example, take $n = 8$, then $(x^{17} - 1)/(x - 1) = 0$ leads to

$$C^8 - 7C^6 + 15C^4 - 10C^2 + 1 + C^7 - 6C^5 + 10C^3 - 4C = 0$$

University of Auckland. H. G. FORDER

73. An example on mathematical induction

In his *Mathematics and Plausible Reasoning* Professor G. Polya has remarked (Volume I, page 119) that a proof by mathematical induction may fail not only if one tries to prove too much but also if one tries to prove too little; and he illustrates the latter possibility by an example on the properties of a recursively defined function. The following example demonstrates the same point and has the added advantage of depending only on elementary school algebra.

If $a_n = \dfrac{1 \,.\, 3 \,.\, 5 \ldots (2n-1)}{2 \,.\, 4 \,.\, 6 \ldots (2n)}$ for n a positive integer, and if $a_k < \dfrac{1}{\sqrt{(3k)}}$ for some positive integer k, then

$$a_{k+1} < \frac{1}{\sqrt{(3k)}} \cdot \frac{2k+1}{2k+2} = \frac{1}{\sqrt{(3k+3)}}\sqrt{\frac{4k^2+4k+1}{4k^2+4k}},$$

from which one obviously cannot deduce that $a_{k+1} < \dfrac{1}{\sqrt{(3k+3)}}$.

On the other hand if $a_k \leq \dfrac{1}{\sqrt{(3k+1)}}$ then

$$\begin{aligned} a_{k+1} &\leq \frac{1}{\sqrt{(3k+1)}} \cdot \frac{2k+1}{2k+2} \\ &= \frac{1}{\sqrt{(3k+4)}} \cdot \sqrt{\frac{12k^3+28k^2+19k+4}{12k^3+28k^2+20k+4}} \\ &< \frac{1}{\sqrt{(3k+4)}}. \end{aligned}$$

It is thus possible to prove directly by induction that $a_n \leq \dfrac{1}{\sqrt{(3n+1)}}$ but not that $a_n < \dfrac{1}{\sqrt{(3n)}}$.

It may also be noted that:

(i) the inductive proof of $a_n \geq \dfrac{1}{\sqrt{(4n)}}$ presents no difficulty;

(ii) as $n \to \infty$, $a_n \sim \dfrac{1}{\sqrt{(\pi n)}}$ by Wallis's formula or Stirling's approximation;

(iii) the above upper bound for a_n can be used in conjunction with the comparison test to prove the convergence of the Maclaurin series for arc sin x at the endpoints of the range $(-1, 1)$ without appealing to Raabe's test.

The upper and lower bounds for a_n can also be used to establish the convergence for $-1 \leq x < 1$ of the series $1 + \sum_{n=1}^{\infty} a_n{}^2 x^n$, the sum of which is a multiple of the complete elliptic integral of the first kind, viz. $\dfrac{2}{\pi}\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{(1 - x \sin^2 \theta)}}$.

A. V. Boyd

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Editorial Note. A very simple example of a result too weak for induction is the inequality $(1 + x)^n \geq nx$, $x > 0$; for from $(1 + x)^k \geq kx$ follows $(1 + x)^{k+1} \geq (k + 1)x - x(1 - kx)$, and $1 - kx > 0$ only if $k < 1/x$. Of course the stronger result $(1 + x)^n \geq 1 + nx$, $x > 0$, may readily be proved by induction.

74. A short evaluation of an integral

We have $(1 + t^4) = (1 + \sqrt{2}t + t^2)(1 - \sqrt{2}t + t^2)$, and, by putting $t = 1/u$,

$$I = \int_0^\infty \frac{dt}{1 + t^4} = \int_0^\infty \frac{u^2\,du}{1 + u^4}.$$

Hence
$$I = \tfrac{1}{2}\int_0^\infty \frac{1 + t^2}{1 + t^4}\,.\,dt = \tfrac{1}{4}\int_{-\infty}^\infty \frac{1 + t^2}{1 + t^4}\,dt$$

$$= \tfrac{1}{4}\int_{-\infty}^\infty \frac{1 - \sqrt{2}t + t^2}{1 + t^4}\,dt, \text{ since } \frac{t}{1 + t^4} \text{ is an odd function of } t,$$

$$= \tfrac{1}{4}\int_{-\infty}^\infty \frac{dt}{(t + \frac{1}{2}\sqrt{2})^2 + \frac{1}{2}} = \frac{1}{2\sqrt{2}}\Big[\tan^{-1}(\sqrt{2}t + 1)\Big]_{-\infty}^{\infty} = \frac{\pi}{2\sqrt{2}}.$$

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CORRESPONDENCE

To the Editor of the *Mathematical Gazette*

DEAR SIR,

The recent growth of interest in mathematics seems to be having the effect of increasing the mathematical sophistication of the general public.

The other day at Victoria Railway Station, for example, I saw the following notice displayed by a bootblack:

$$\boxed{\text{sir} + \frac{\text{sir}^3}{3!} + \frac{\text{sir}^5}{5!} + \ldots ?}$$

Pondering the fellow's hyperbolic erudition, I made my way to the bus station, and was pleased to see that my bus had just drawn in. Boarding it, I heard a woman behind me exclaim, in a tone which suggested that the means of solving some problem or other had at that moment dawned on her:

"Get to the back of the CUBE root!"

The emphasis on the penultimate word indicated that until then she had been getting to the back of the square root. For my part, I must confess that while I am able to *extract* square, cube or other roots, the process of "getting to the back of *a* root" is unknown to me. The woman subsequently boarded the same bus, and I would have asked her to enlighten me, but unfortunately the bus was crowded and I was on an inside seat while she was standing up; moreover she wore a somewhat disgruntled expression.

Alighting from the bus outside the National Portrait Gallery, I was amused to see that among the usual half-dozen or so pavement artists with their coloured landscapes was one who had chalked nothing but these words:

Modern art
Gets worse and worse
So I'm resolved
To write $\tanh^{-1} Q$.

Of course I could not refrain from dropping a coin into his cap and so earning his proffered gratitude.

Yours etc., BASIL MAGER

123 *Chanctonbury Road*
Burges Hill, Sussex

To the Editor of the *Mathematical Gazette*

DEAR SIR,

Mr. H. W. Clayton of Summer Fields Oxford, will, on behalf of S.A.T.I.P.S.—the Society of Assistants Teaching in Preparatory Schools —be running a Mathematics Conference in York, January 4 to 6, 1962. A similar Conference was held in Cambridge in 1960 and many valuable contacts were made. The chief aim of these Conferences is to enable Public Preparatory school mathematics masters to meet and to discuss difficulties, methods, syllabus and such things, and to help people to realize that there is something beyond Common Entrance. At Cambridge, there were about 25 masters from Public Schools and about 100

from Preparatory schools, including some Head-Masters. Many Preparatory School Masters are members of the Mathematical Association and S.A.T.I.P.S. would welcome Public School masters as Associate members of their society for an Honorary Membership Fee of 5/- a year. Expenses for Conferences are of course extra, usually about 4 guineas, for accommodation and meals. Details of Membership may be obtained from J. B. Maplin Esq., The Pound, Blatchington, Seaford, Sussex. He will of course, send further details of Conferences to members.

Yours etc., JOHN WILLIAMS

Little Thorns, Gatton Point
Redhill, Surrey

To the Editor of the *Mathematical Gazette*

DEAR SIR,

Readers of the *Gazette* may be interested to have some information about Mathematical competitions, which have gained increasing support, in recent years, in the Soviet Union and the U.S.A. Usually they have a twofold object—to detect emergent mathematical ability and to stimulate interest in the subject. The Eötvos Prize competition, which has been held annually in Hungary from 1894 to the present time (with minor exceptions) is the classic example, and it has been remarked that Hungary has produced many outstanding mathematicians, several of whom were Eötvos prizewinners. [See the articles referred to below.] The Russian Mathematical 'Olympiads' began in 1934 in Leningrad; today about ten different competitions are organised by the University centres. Some 1000 students aged 14–18 are annually involved in the Moscow Olympiad. Several Mathematical Contests are currently held in the U.S.A.—of these the best known are the Putnam competition (at undergraduate level), the Stanford University competition conducted by Prof. G. Polya, and the National High School Contest sponsored by the Mathematical Association of America. This latter contest, of which some details are given below, involves only basic algebra, geometry and trigonometry (i.e. O level with minor exceptions caused by syllabus differences). In 1960 some 150,000 students in American and Canadian High Schools took part. Further information about mathematical competitions is contained in articles in the *American Mathematical Monthly*, March 1959 and May 1960 and in *Mathematics Teacher*, December 1958.

Details of the National High School Contest

This is a multiple choice test lasting 80 minutes in which wrong answers are penalised so that random choice would produce zero score. Marking is simple and standardised. Outstanding individual performances are published and several medallions awarded. The three best papers in any participating school are totalled to give a 'team score' though the Contest is not envisaged as an inter-school competition. Coded results are published in order that participating schools may discover their relative status. The test usually consists of 40 questions, arranged in increasing order of difficulty. Some specimen questions follow (1960).

(4) Each of two angles of a triangle is 60° and the included side is 4 inches. The area of the triangle, in square inches, is:

(A) $8\sqrt{3}$ (B) 8 (C) $4\sqrt{3}$ (D) 4 (E) $2\sqrt{3}$

[The letter of the correct answer, 'C', is to be written in a space on the answer sheet.]

(17) The formula $N = 8 \,.\, 10^8 \,.\, x^{-3/2}$ gives for a certain group, the number of individuals whose income exceeds x dollars. The lowest income, in dollars, of the wealthiest 800 individuals is at least:

(A) 10^4 (B) 10^6 (C) 10^8 (D) 10^{12} (E) 10^{16}.

(27) Let S be the sum of the interior angles of a polygon P for which each interior angle is $7\frac{1}{2}$ times the exterior angle at the same vertex. Then

(A) $S = 2660°$ and P may be regular (B) $S = 2660°$ and P is not regular (C) $S = 2700°$ and P is regular (D) $S = 2700°$ and P is not regular (E) $S = 2700°$ and P may or may not be regular.

(34) Two swimmers, at opposite ends of a 90 ft. pool, start to swim the length of the pool, one at the rate of 3 feet per second, the other at 2 feet per second. They swim back and forth for 12 minutes. Allowing no loss of time at the turns, find the number of times they pass each other.

(A) 24 (B) 21 (C) 20 (D) 19 (E) 18.

(39) To satisfy the equation $\dfrac{a+b}{a} = \dfrac{b}{a+b}$, a and b must be:

(A) Both rational (B) both real but not rational (C) both not real (D) One real, one not real (E) one real, one not real or both not real.

It has been suggested that some schools in this country might like to participate in this competition. I should be glad to supply further information to anyone who is interested.

Yours etc., F. R. Watson

Manchester Grammar School

To the Editor of the *Mathematical Gazette*

Dear Sir,

May I congratulate N. de Q. Dodds on discussing the matter of elementary division and the method of setting it out? While not sure that he has the answer as regards setting out, I am convinced that some reform is most desirable. It is extremely confusing to a poor pupil to find that sometimes the divisor is on the left, $23\overline{)4187}$, sometimes on the right, $4187 \div 23$ and sometimes underneath $\dfrac{4187}{23}$. No wonder pupils will write $4187 \div 23$ or $23 \div 4187$ indiscriminately.

Some people think that if a pupil is so poor that at the age of 13 or so he is still confused about division, then one should not bother about him (or her). But it is quite possible in this country for girls who are poor mathematically to train as primary school teachers and thus pass on their own confusion.

Can anybody suggest for long division a method of setting out which does away with this confusion? The weakness of N. de Q. Dodds' method (p. 181) is that the divisor is placed too far away from the working.

There seem to be two lines of approach:

(1) Abolish the ÷ sign and set down division by the fraction $\frac{18}{3}$ or the half-bracket 3)18. No change in setting out of long division would be needed, but I agree with Mr. Dodds that "divide by" is better than "divide into."

(2) Abolish the 3)18 method. A method would have to be devised for setting out long division so that the divisor is either on the left or underneath the dividend. Can anybody devise such a method?

Yours etc., UNA DROMGOOLE

Riccarton H. S., Chch. N.Z.

To the Editor of the *Mathematical Gazette*

DEAR SIR,

Dr. Easthope says in his letter published in the Mathematical Gazette for December 1960 that one cannot always impose *real* frictionless constraints appropriate to Bertrand's Theorem. Would he accept a massless structure as *real*? Since a frictionless constraint is really the idealisation of a constraint of low friction and is accepted as *real*, a massless structure, the idealisation of a structure of small mass, should, one would think, also be accepted as *real*. If massless material is allowed it is perfectly possible to provide constraints which will allow motions as close as we please to the free motion. Thus his example, the simple rod whose instantaneous centre is outside the rod, may be provided with a massless link. One end of the link is pinned to any point of the rod, the other to any point in space. By choosing the latter point at or near the centre of rotation of the final free motion one may obtain constrained motions identical with the free motion or as close to it as one pleases.

Massless constraints of this kind do no work in a small displacement of the system. They can only acquire energy by attaining infinite velocity; and this they cannot do because of the finite velocities of the massive bodies to which they are attached. Such massless constraints are in many cases—probably in all cases for which the initial state is one of rest—equivalent to constraints not involving massless bodies. For example, the massless link just described exerts the same constraint (for two-dimensional motion) as a frictionless peg attached to the rod and sliding in a circular slot cut in the plane on which the rod is resting.

The constraints considered in Bertrand's Theorem must be compatible with the initial motion. To satisfy Dr. Easthope's criterion they must also be capable of variation so that "the constrained motion differs by as little as one pleases from the motion of the free system." Since the instantaneous motion of a rigid body is simply a screwing motion about some axis, a massless constraint compatible with this may obviously be applied to any rigid body of the system. This constraining structure may then be carried as a whole on another structure which permits

rotations about and sliding along an arbitrary axis. By choosing this latter axis close to the screw axis of the final free motion of the same rigid body, we can produce constrained motions as close as we please to the free motion. Thus satisfactory *real* constraints can be provided. The kinetic energy of the free motion is therefore equal to the maximum of the energies of possible constrained motions; and is a stationary value of these energies. Incidentally, because of an error of sign, the formula in Dr. Easthope's letter shows the free energy to be a minimum.

Yours etc., A. J. HARRIS

21 Windmill Court Windmill Road
Ealing, London, W.5.

REVIEWS

A Companion to School Mathematics. By F. C. BOON. Pp. 302. 30s. 1960. (Longmans, Green and Co. Ltd.)

In 1903 F. C. Boon received the princely salary of £80 for his year's work as tutor at Trinity College, Carmarthen. The College records give no other appreciation of the work done by Mr. Boon but if this book is any indication of the life and vitality of the teaching of the man who later became the Principal Mathematical Master at Dulwich College then the Students of his period were very fortunate.

This book was first published in 1924 and was a treasured possession of the older generation of teachers. Now it has been re-printed with a Foreword by Mr. A. P. Rollett and a new Bibliography.

Students of Mathematics have text books to keep them on the narrow examination route. They need something else as well. The exploration of byways which make all the difference between arid, pure knowledge and the more rounded, happy experience of the student who can see the power and ramifications of ideas, their origin and the very human beings who originated them. We have all tried, or wanted to try, to do this but often, more particularly in the early days of teaching, failed to get the time, the sources and dare I say it, sometimes the inspiration.

Without going into details (See the review by Mr. C. O. Tuckey, The Mathematical Gazette, Vol XII No. 175, March 1925) here is just the book. It should be in the possession of every teacher of Mathematics and available in every library, and this includes Public Libraries. Certainly every Senior boy in School and Training College Student should be offered the opportunity of dipping into it.

It has a short but excellent bibliography of books of a kindred spirit which should also be in every library.

It is well printed and bound and likely to withstand the eager wear and tear that should be its lot.

D. D. REES

Mathematical Topics for Modern Schools. By E. J. James. Books 1–3. Pp. 16, 1s. 6d. each. 1961. (Oxford University Press)

Mr. James has produced an excellent set of books for use in Modern Schools. Although grouped in sets of three for each of the four years of the (usual) school life of most modern school children, they are in no sense meant to be used as text books. They should form a fruitful source of mathematical ideas and topics for both teachers and children. Teachers in junior schools should also find much to interest their pupils, but the fact that so much of the work is already covered in most good junior schools is one of the limitations of these books for use in secondary schools. Teachers will be particularly interested in Book II, Year II, in which examples are given of curve stitching.

D. M. Mowat

Daily Life Mathematics Book V. By P. F. Burns. Pp. 340. 15s. 6d. 1961. (Ginn and Co.)

The fifth book in the series Daily Life Mathematics will be welcomed by teachers in Secondary Schools, some because they have used the earlier books and are convinced that the order and method followed is the right one; others because they like to use the books as extra text-books to study certain topics or to give illustrations of the applications of Mathematics to every day situations.

The author's claim that Book V "more than adequately covers the ordinary G. C. E. syllabuses" is justified but most teachers would have liked similarity to have been included in the text.

The course is unified as there is no major division into arithmetic, algebra and geometry but the arrangement is such that the pupil can easily grasp the fundamental principles common to them all. The underlying idea is to teach the pupil to think mathematically and in this the author succeeds.

D. M. Mowat

The Story of Numbers. By Thyra Smith. Illustrated by F. T. W. Cook. Set of four books. Pp. 16, 1s. 9d, each book. Library Edition 8s. 6d. 1961. (Blackwell, Oxford)

This set of books gives the story of reckoning simply, from 'matching' to the employment of calculating machines. The important points in the development of calculation are placed in their historical and social context. Children will see the reasons for the varying ways of writing figures and setting out calculations, from the mere placing of strokes or pictorial representation to the modern Arabic notation. The books are attractively illustrated.

G. A. Moss

Notes and Exercises on Coordinate Geometry. By I. TOMLINSON. Pp. 128. 9s. 1959. (Methuen)

This little book covers a surprising amount of ground in a small space. This is achieved by the reduction of bookwork to a minimum, what there is being concise and systematic. The bulk of the text consists of drill exercises, but there are also three sets of carefully graded miscellaneous examples, and a good selection of examination questions from papers at both Ordinary and Advanced Levels. The author's methods are sound and his exposition is clear and up-to-date in approach (parameters are freely used) and layout. The range of the book extends just as far as pole and polar, and stops rather short for some A Level syllabuses in Pure Mathematics, but the author only claims sufficiency for "Mathematics for Science at A Level" and for that it is fully adequate. The arrangement of the topics is such that it is extremely easy to find what one wants; and I have pleasure in recommending this book as, within its self-imposed limits, a thorough and agreeable work.

A. R. PARGETER

Elementary Coordinate Geometry. By C. V. DURELL. Pp. 341 + xxiii. 17s. 6d. 1960. (Bell)

A new text-book by C. V. Durell practically reviews itself: those who are already familiar with his books will expect—and find—mastery of subject-matter, clarity of exposition, thoroughness of detail, and a wealth of well-chosen and carefully graded examples. In fact so thorough is the treatment that it provokes my one criticism: all but the really able pupils will be in danger of being over-whelmed; it is not a book for any but those students who intend to take their mathematics seriously.

The subject is developed on traditional lines, with the circle being the first curve treated, but this chapter is comparatively brief and deals only with those topics that display the advantages of the analytical method (and the author does in fact suggest that this chapter may be postponed). Prior to this, line-pairs through the origin receive brief mention, but full treatment of line-pairs is reserved for Part II. As one would expect in an up-to-date work parametric methods are used freely throughout; the basic ideas have a chapter to themselves, after which the properties of the parabola and rectangular hyperbola are developed fully. The ellipse is regarded in the first instance as the orthogonal projection of a circle, and the focus-directrix property derived, and not used as a definition. It is refreshing to find a parallel treatment of the hyperbola, in which the general hyperbola is defined as the orthogonal projection of a rectangular hyperbola. Advanced Level requirements up to confocal conics, which are just touched upon, are covered in Part I, which is available separately. Part II, for the Scholarship candidates, introduces the use of determinants in an informal fashion, and discusses —to pick a few topics at random—the general line-pair, pole and polar (already mentioned in Part I but here treated more fully), oblique axes, polar coordinates, the general equation, homogeneous coordinates (cartesian), line coordinates, systems of conics; and ends with a short

chapter on curvetracing, most of the examples being parametric in form.

Throughout the book the use of geometrical methods when suitable is encouraged; whatever the advanced worker in *algebraic* geometry may think, it is important to remember that from the elementary stand-point the subject-matter of *coordinate* geometry is, after all, geometry rather than algebra. The beginner often forgets this, or even imagines that it is not quite the thing to invoke pure geometry—let alone mix the two, which the author is quite happy to see done.

There are extensive reference lists of formulae, an index (still too rare a feature of elementary texts), and for revision, ample Test Papers, besides some useful Quick Revision Papers for rapid testing of knowledge of formulae and methods.

A. R. Pargeter

Elementary Mechanics. Volume 2. Quadling and Ramsey. Published—G. Bell and Sons Ltd., 1959. Price 17s. 6d. 642 pages.

This book is designed to cover the work in Mechanics for A Level, The Mechanical Sciences Tripos Qualifying Examination at Cambridge and the first year examinations of many Engineering Degree Courses. The high standard of Vol. 1 has been fully maintained, but the appeal of this book is more specialised. The less able Science Sixth Formers would find it difficult, but for the Mathematics Specialist or the best Science students it is an excellent book.

One of the interesting features of the book is the free use made of vector methods, which was foreshadowed in Volume 1. Differentiation of vector functions of time is introduced in the discussion of circular motion in the second chapter, and vector methods are used to obtain the radial and transverse acceleration components for circular motion with variable speed and again later in dealing with the general problem of velocity and acceleration in two dimensions. Whenever appropriate vector methods have been used in solving problems, and the introduction given here should prove of great value to students going on to more advanced work.

Another point of interest is the use made of numerical methods in integration, both in the text and in examples.

The book also includes discussion of Oscillation, motion of Rigid Bodies, Internal Stresses in beams and stresses in Light Frameworks, Mechanics of Fluids, Constrained Motion in two dimensions, motion in a Resisting Medium, motion of interacting bodies, and application of the Energy Principle.

The text is clear and well arranged, worked examples are well chosen, and diagrams throughout are excellent.

Examples are plentiful, well graded and interesting, and three sets of Miscellaneous Examples are included. Answers are provided at the end of the book.

F. E. Chettle

Discovering Mathematics. By H. A. SHAW and F. E. WRIGHT. Edward Arnold, Ltd. 1960. Book 1. 224 pp. 9s.

The title of this book suggests a valuable and rewarding approach to the teaching of mathematics in Secondary Schools. As the authors say in their preface, "in the past there has been far too much mechanical teaching." "So often the mathematics in the Secondary Modern Schools has consisted of pounding away at material experienced in the Junior School."

The book is divided into sections on Geometry, Algebra and Arithmetic, the teacher being expected to choose his own sequence of topics. This needs some care. For example, on P. 135 we find $\pi = \frac{C}{d}$ $\therefore$ $C = \pi d$, whereas the necessary algebraic ideas do not occur until later in the book. There is a welcome emphasis on applications to surveying. Some historical material is included. The book is attractively illustrated and the diagrams are bold and clear. Unfortunately, there are some errors in the diagrams. For example, there is insufficient data given for Q.3 on P. 63 and for Q. 11 on P. 122, and some of the other diagrams need free interpretation. The book ends with revision examples and tests.

It is a pity that the book cannot always be said to live up to its title. There are examples of rules being quoted with little, if any, justification, e.g. in the work on fractions. The chapter on simple equations has, in the seventh line: "We use the rule that when we change the side we change the sign." There are also some inaccuracies. On P. 154, £8 — £14 should equal $-$£6. Lines 3–5 on Page 156 are linguistically meaningless.

Despite these blemishes, its many attractive features make the book worth consideration by those teaching in Secondary Modern Schools.

N. P. PAYNE

Preliminary Mathematics for the Craft Apprentice. By TREVOR J. ROGERS and GORDON TAYLOR. Edward Arnold, Ltd. 1960. 224 pp. 9s.

This book is intended for use by the middle streams of the Fourth Year in Secondary Modern Schools and for the initial courses in Evening Institutes and Colleges of Further Education. The material has been confined to that required for such examinations as the Preparatory Craft Course of U.L.C.I., the J.1. Course of City and Guilds and the Introductory Technical Course A of the Union of Educational Institutions. This explains the absence of certain topics, such as elementary trigonometry and logarithms, which one might expect such pupils to have met before leaving school.

A difficulty in writing a book for use in the later stages of the school is in deciding how much early work to include. The authors, perhaps wisely, provide chapters and exercises to cover the whole of these syllabuses.

There is a plentiful supply of graded exercises, with an emphasis on the practical applications of the work to everyday situations and to

engineering. An interesting feature is the placing of drill exercises sometimes *after* the practical type of questions. The explanations in the book-work are good and the authors are at pains to explain the processes being used. For example, it is unusual in a text-book to find an explanation for the process of extracting a square root.

The book can be recommended for consideration by those providing courses for would-be apprentices to any branch of engineering, whether in School or College.

N. P. PAYNE

Introduction to Higher Mathematics. By CONSTANCE REID. Pp. 184. 12s. 6d. 1960. (Routledge and Kegan Paul Ltd.)

This is another delightful volume 'for the general reader' by the author of *From Zero to Infinity*; in this rather more ambitious work we have another glimpse of number theory, a brief introduction to the concept of group, something about the calculus, transfinite numbers, non-Euclidean geometry, n-dimensional geometry, topology, and sentence logic. As in her previous volume the author seeks to convey the spirit of mathematics by showing the difficulty that may lie in answering seemingly simple questions. In some sections quite difficult arguments are presented, for example the division of a circumference into an infinity of congruent sets, but in others only the simpler concepts are introduced, for instance in the section on groups (where an unfortunate confusion of addition with multiplication on page 4 mars the account of closure under the group operation). The section on calculus makes little effort to help the novice, employing from the outset the traditional Δx, Δy notation. In the excellent final chapter on decision methods it is stated (on p. 128) that elementary arithmetic is a decidable theory, but this is true only if elementary arithmetic is taken to mean arithmetic with addition but lacking multiplication (or *vice-versa*).

R. L. GOODSTEIN

Mathematics in the Making. By L. HOGBEN. Pp. 320. 50s. 1960. (Macdonald)

The use of the historical method in the teaching of mathematics takes many forms, from a slavish imitation of the development of the concepts, to a judicious sideglance at the historical setting in which the concept evolved; at one extreme the student is denied the simplification which time brings to the presentation of new ideas and at the other extreme the historical details are merely an additional burden on the memory. Hogben avoids both these dangers and has succeeded in producing an account of the development of elementary mathematics which makes interesting and important contacts at many points with world history. The great feature of the book is the huge collection of magnificent diagrams, visual aids of a quality that has never been surpassed. One can but envy the author the service his artists and Editor have given him. Unhappily the text does not always achieve the same high standard.

Aside from technical mistakes, like the definition of the derivative on pages 217, 218 as the limit of

$$\frac{f(x+\frac{1}{2}\Delta x)-f(x-\frac{1}{2}\Delta x)}{(x+\frac{1}{2}\Delta x)-(x-\frac{1}{2}\Delta x)}$$

as $\Delta x \to 0$ (which gives $|x|$ a zero derivative at the origin), and the assumption (p. 240) that a function is necessarily equal to its Maclaurin expansion, there is a marked failure to appreciate the importance of work like Euclid's theory of proportion and his invention of the axiomatic method, and the mathematical significance of such theorems as the impossibility of trisecting an angle by ruler and compass constructions, a failure that is a product of the author's preoccupation with the social significance of mathematics.

R. L. G.

Complex Numbers. By W. Ledermann. Pp. 62. 5s. 1960.

Partial Derivatives. By P. J. Hilton. Pp. 54. 5s. 1960. (Routledge and Kegan Paul)

Two welcome additions to the new *Library of Mathematics.* A crisp treatment of complex numbers including the exponential and circular functions, with a very careful account of angle, and a useful collection of examples. It is perhaps unfortunate that Log z, with a capital letter, is used for the principal value of the logarithm in opposition to the notation recommended by Mathematical Association Reports for many-valued functions. Two small misprints are "i" for "$\frac{1}{2}\pi$" on page 24 and "y" for "Y" on page 27.

The volume on *Partial Derivatives* is a sequel to the author's *Differential Calculus* in the same series and achieves a high standard of clarity and precision, with many illuminating examples. Topics discussed include Jacobians and maxima and minima.

R. L. G.

Introduction aux Mathématiques Modernes. By A. Monjallon. Pp. 180. 2000F. 1960. (Vuibert, Paris)

The topics considered in this easy-to-read little book include the algebra of sets (and of relations) sentence logic (truth tables and a hint of proof theory), the language of quantification logic, and a few elementary properties of commutative groups.

R. L. Goodstein

Einführung in die Funktionen Theorie. By I. I. Privalov. Part III. Pp. 186. DM 12.90. 1960. (Teubner, Leipzig)

This final volume of Privalov's Introduction to Function Theory includes elliptic functions, conformal mapping and some extremal problems. The discussion of conformal mapping is, as one would expect from the first volume, extremely good, clear and comprehensive.

R. L. G.

Complex Variables and Applications. By R. V. Churchill. 2nd Ed. Pp. 296. 52s. 6d. 1960 (McGraw-Hill)

The first edition of this book was warmly welcomed by Prof. J. L. B. Cooper in his review in the October *Gazette* 1949 (p. 220). This new edition is considerably enlarged (and is much more expensive); amongst the changes which have been made is the use of more modern terminology, the inclusion of proofs of results previously stated without proof, a new chapter on Integral Formulas of the Poisson Type, new material on analytic continuation, and the behaviour of functions on the neighbourhood of an essential singularity. The number of examples has been substantially increased.

R. L. G.

Elementary Pure Mathematics. By J. D. Hodson. Pp. 314. 17s. 6d. 1960. (Macmillan and Co.)

The author is Senior Mathematical Master of Kingswood Grammar School, near Bristol. His book covers the Pure Mathematics syllabus for most Advanced level examinations taken by scientists. Each of the six chapters contains some work on geometry and the usual four analytical divisions, the sequence and arrangement of the subject matter being determined by the author's purpose of coordinating the analytical work with an early development of the calculus.

This is a stimulating and invigorating book. The author's sixth form teaching experience has enabled him to arrange a concentric course very successfully, giving us a rapid initial development over a wide field, plenty of concise yet scholarly material with well interspersed sections of bookwork and examples. Each chapter is intended to represent 4 to 6 weeks' work with a mathematical sixth form, or perhaps double that time with scientists. The text is clear and attractive, there is no padding or trivial matter, and interesting matter continues to appear right to the end of the book.

A detailed list of the contents of the chapters, the relevant text diagrams, test papers, answers and an index are all provided in excellent style.

J. W. Hesselgreaves

Intermediate Pure Mathematics. By J. Blakey. Pp. 458. 21s. Second edition. 1960. (Cleaver-Hume Press)

The second edition adds some seventy pages, the extra subject matter being: Polar Coordinates, Logarithmic, sine and cosine series, Coaxial circles, Determinants, Curvature, Differentiation of inverse trigonometrical functions, Integration by partial fractions and by substitution, Mean values, Integration by parts, Differential equations. There are now some 900 examples, mainly taken from former Advanced and Scholarship level papers. Although there is no formal work on geometry, there are 100 examples on plane and solid geometry.

The first section of the book follows the traditional plan. After five chapters on algebra come two each on trigonometry, coordinate geometry and calculus. Then there follows the new material mentioned above. Bookwork is given very fully and covers everything a student is likely to need. A special feature of the book is the wealth of illustrative examples, in which the manipulation is set out in great detail. This makes it very suitable for a class book when the student has only occasional contacts with the teacher, or as a supplementary reference textbook.

The printing is good, if a little solid, the diagrams are clear, a good index is provided, and the answers to the examples are given both chapter and page references.

J. W. H.

Les Structures de Commutation A m Valeurs et les Calculatrices Numeriques. By MIDHAT J. GAZALÉ. Pp. 76. 14 NF. 1959. (Paris: Gauthier-Villars; Louvain: E. Nauwelaerts)

Electronic computing machinery is generally constructed from devices which are *bistable*, i.e. which store information by assuming at any moment one of two possible stable states, and a natural mathematical notation in which to express the properties of such devices is Boolean algebra. M. Gazalé attributes this limitation to two states primarily to the comparative unreliability of multistable devices, but he foresees reliable multistable devices as a technological possibility, and accordingly he argues the desirability of developing algebraic techniques similar to those of Boolean algebra for the expression of their properties. His concern, then, is with a general finite set E and the set of all functions of one or more variables defined over E and having values in E. But since he is addressing himself to engineers as much as to mathematicians, he develops a diagrammatic notation of networks and gates (*assemblages* and *combinateurs*), which he uses to assist the mathematical argument in a manner reminiscent of the use of diagrams in geometry.

A central problem in this field is the discovery of sets of functions which are *functional*, in the sense that all the other functions can be derived from them by iteration. M. Gazalé recalls, in a useful historical section, various functional sets of functions discovered by earlier writers, and shows how it is possible to generalise some of these results. In particular, he generalises the discoveries of Webb and Post of functional sets consisting of a single function and a pair of functions respectively.

A chapter is devoted to the important special case in which the set E contains a prime number of elements, which can therefore be regarded as elements of a prime number field. Here, considerable use can be made of standard algebraic techniques, and M. Gazalé develops a useful method for finding a polynomial in several variables, given its values at all points. Like many other writers on this subject, however, he overlooks the immediate extension to prime-power fields.

Besides presenting a number of interesting results, this book contains a collection of useful algorithmic tricks, but it is a pity that the text is marred by a number of misprints.

R. A. CUNINGHAME-GREEN

Cours de Mathématiques Générales; Tome IV. By R. GARNIER. Pp. vi, 275. 1959. 45 NF. (Gauthier-Villars, Paris)

This book provides an excellent undergraduate course in ordinary and partial differential equations, including the theory of Fourier series. The theory is clearly explained, and many examples are worked out.

The book concludes with a large collection of problems in analysis.

E. J. F. PRIMROSE

A Course in Pure Mathematics. By MARGARET M. GOW. Pp. xi, 619. 40s. 0d. 1960. (English Universities Press)

This book covers the new syllabus in Mathematics for Part I of the London B.Sc. General Degree and in ancillary Mathematics for Honours courses in scientific subjects. All the usual topics in algebra, coordinate geometry and calculus are covered. A good standard of rigour is maintained, but the emphasis is mainly on the mastery of techniques.

I have only a few minor criticisms. Firstly, on p. 14 it is stated that every equation $f(x) = 0$, where $f(x)$ is a polynomial, has at least one root. Even with the qualification further down the page that the roots of the equation $f(x) = 0$ may not be real, this may be puzzling to anyone who has not met complex numbers (which do not appear until later in the book). Secondly, at this level it should not be stated (as on p. 191) that the tangent to the curve $y = f(x)$ at a maximum or minimum point is parallel to the x-axis. For the curve $y = x^{2/3}$ at $x = 0$, the tangent is the y-axis, yet the origin is a minimum point. Thirdly, fig. 101(b) on p. 494 has been wrongly drawn: the inner circle should have radius $\frac{1}{4}a$, not $\frac{1}{2}a$.

This book will be very useful for the students for whom it is intended, particularly those who have to work without much supervision. There is a fine collection of examples, 350 worked and 1550 to be solved by the student.

E. J. F. PRIMROSE

A Course in Applied Mathematics. By D. F. LAWDEN. Pp. xv, 655. 1960. 70s. 0d. (The English Universities Press Ltd.)

This is an important book and it will command the attention of those concerned with the preparation of students reading Applied Mathematics for the B.Sc. General Degree. It is divided into four parts which together constitute a generous account of Dynamics, Statics, Field Theory (including the electro-magnetic field) and Hydromechanics rather more than is required for the Part II London University General Degree and equivalent examinations. It is not written for the mathematical specialist but it assumes the reader has attained the standard of Advanced Level G.C.E. and is acquainted with elementary vector theory. The author's aim, he says, is to present "a complete logical structure. I have not therefore avoided difficult passages in the development of the subject, either by ignoring them, appealing to the reader's intuition or by offering an argument by analogy."

But it should be said at once that the students' needs—including those of part-time students at technical colleges—are evidently considered to be paramount. There is a minimum demand on knowledge of pure mathematics (gradually increased since the average reader will be taking a pure mathematics course), there is a large number of illustrative solved problems, each of the kind set in university examinations, with many more similar problems for use as exercises. The author is careful, clear and skilful in presenting the theory and to a great extent achieves his main aim. The result is a book which students will find easy to read and sufficient for their immediate needs. There is hardly any room for a misunderstanding and the student should soon learn to solve examination questions.

The treatment is for the most part the traditional one both in concept and technique, and one wonders whether, even within the framework of the General Degree syllabus, it would not have been advantageous to introduce more new concepts and techniques and concentrate less on example solving. In rigid dynamics, for example, where the orthogonal transformation is so prominent, matrix calculus is the ideal method. It may, of course, be thought that this would place too heavy a burden on the student. But the reviewer has the feeling that the great merit of this work, namely consideration of the student, is also its weakness. A complete logical structure is the wrong aim—the students, one fears, will not read any other books, and if they do, they will find the going harder than it need have been.

E. V. Whitfield

Calculus and Analytic Geometry. By G. B. Thomas. 3rd edition. Pp. xii, 1010. 1960. 53s. 0d. (Addison-Wesley)

The first two editions of this book were reviewed in the Gazette, Vol. XXXVII, pp. 160, 224. The present edition is even larger, and excellent value for the price.

E. J. F. Primrose

Pure Mathematics. By F. Gerrish. Vol. I—Calculus. Vol. II—Algebra, Trigonometry, Coordinate Geometry. Pp. xxv, xxix, 758, 48 (answers). 1960. 25s. 0d. and 35s. 0d. (Cambridge University Press)

Since the new regulations for the General B.Sc. Degree of London University came into force, there has been no book which covers the whole syllabus for Pure Mathematics in Part I. The present book has been written primarily to satisfy this requirement, but it will also be useful for other purposes. Students who take a one-year subsidiary course of mathematics will find the book very useful, and any boy or girl who is spending a third year in the sixth form before going on to a university will derive great benefit from it.

One of the many virtues of the book is that the treatment is rigorous without being difficult to follow. In fact, the style of writing and the clear presentation of the material make this an easy book to read. The author makes sure that there will be no gaps in the reader's knowledge by starting each main branch with work which should already be familiar.

The examples should be sufficient for even the most industrious student. They include many results of considerable interest for which there is not room elsewhere.

I can strongly recommend this book to anyone who wishes to pursue Pure Mathematics beyond 'A' level.

E. J. F. Primrose

Intermediate Algebra. By R. Dubisch, V. E. Howes, and S. J. Bryant. Pp. xi, 286. 36s. 1960. (New York: John Wiley and Sons; London: Chapman and Hall)

Intermediate Algebra for Colleges. 2nd Edit. By J. B. Rosenbach, E. A. Whitman, B. E. Meserve, and P. M. Whitman. Pp. xviii, 306. 1960. (Ginn and Co.)

Both these books are carefully written, and are presented in a clear and attractive way. They are designed for American students of college level who have a limited background of high-school algebra, and they cover the traditional topics for the American 'Intermediate Algebra' college course. They assume a certain maturity of mind in their readers, and are written so as to look forwards towards modern mathematics on the one hand, and yet backwards towards the student who knows very little algebra on the other hand. They assume an elementary knowledge of algebra in the sense that they are not suitable for beginners, but nevertheless they begin their treatment from fundamentals.

Both books probably play a useful role in the contemporary American pattern of mathematical education, but it is difficult to see what function they could perform in Britain. Sometimes they seem to cut right across British traditions and methods. For example in Dubisch's book quadratic equations are not introduced until Ch. 6, p. 119, but the double subscript notation and determinants appear in Ch. 5. ($a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ etc.). In discussing the double subscript notation Dubisch addresses the student thus: "A change in notation is desirable, however, not so much because it is needed in the simple case, but because it is used extensively in more advanced work. If it is learned now it will be available for your use when needed." The spirit underlying this statement seems contrary to that motivating most British mathematics teachers, who usually prefer to use and justify a procedure as and when it becomes necessary. A similar order of treatment occurs in the Rosenbach book where, for example, the binomial theorem and complex numbers are treated before logarithms.

It is difficult to see who, in this country, would actually use these relatively expensive books. Their chief value might be to teachers who would appreciate their clarity and style, and who would be given an insight into the possibility of using different methods of treatment and order of presentation. Both books adopt the practice of giving answers to the odd numbered questions only, so that students and teachers have the choice of working with or without answers. This practice might be adopted more often by writers of British text-books.

Jack Wrigley

Science, Mind and Method. By R. W. Harris. Pp. 116. 9s. 6d. 1960. (Blackwell, Oxford)

This is a collection of short extracts from classical works of science and methodology. Each extract is headed by a brief account of the life and achievements of its author. So vast a historical range of science is touched upon that the individual extracts can hardly be regarded as more than guides to further reading. As a skeleton for a course this collection seems quite adequate. It is too thin to serve as an independent text.

R. Harré

General Degree Applied Mathematics. By S. L. Green and J. E. C. Gliddon. Pp. 346. 18s. 1959. (University Tutorial Press Ltd.)

With skill and experience akin to that of a medical practitioner, the authors have diagnosed the requirements of students reading applied mathematics for the General Honours degree of London University, and produced a first-class tonic. Together with Mr. Green's *Dynamics*, the book aims at covering the syllabus for the studies mentioned above, with the exception of statistics, and the only previous knowledge of the subject demanded is Advanced Level Applied Mathematics. The prescription is made up of eleven chapters ranging from Vector Algebra and Vector Analysis to Electrostatics and Magnetostatics; and it is all so closely allied to the syllabus that one wonders whether eighteen shillings is not a remarkably cheap price to pay for non-attendance at lectures.

To me, the most striking part of the book is the chapter on the principle of Virtual Work. Here is a genuine attempt to put into print, as clearly as possible, what an experienced lecturer would do and say in the class room, and with worked examples on each section. This could well resolve most of the difficulties the student meets in this topic.

Chapter II, on Vector Analysis, has all the theorems necessary for the latter part of the book, but from a field theory point of view, I would have placed it just before those final chapters on Hydrodynamics, Gravitation and Electrostatics and Magnetostatics. And further, I would have liked more stress placed on the viewpoint that these topics are examples of vector fields, satisfying their own particular differential equations, and governed by various boundary conditions. In this way the student would, I believe, be more stimulated and see the broader picture more easily. But simply from an examination point of view, the actual presentation, though a little piecemeal, should be perfectly satisfactory. No tonic can have an everlasting effect.

There are numerous worked examples throughout the book and, in brief, I would say that future examiners on this syllabus will not find it easy to set new questions to test the ingenuity of the good student, whether they are on bending beams or hydrodynamics, normal modes or the vibrations of a uniform string. This book will probably become a best seller within the appropriate student population.

R. Buckley

The Theory of Matrices. By F. R. GANTMACHER. Two volumes, pp. 374 and 276. 1959. (Translated from the Russian by K. A. Hirsch; Chelsea Publishing Company, New York)

In the last two decades Soviet mathematicians have produced a series of remarkable books, whose common feature is the stress laid on thoroughness and intelligibility rather than on slickness of presentation. Professor Gantmacher's two volumes fall into this category, and readers in English-speaking countries owe a debt of gratitude to the translator and the publishers for making available to them this important treatise.

The work provides not so much a systematic development of linear algebra as a detailed study of a number of special topics in matrix theory, an account of some of which is not easily found elsewhere. However, the discussion is based firmly on the general theory of matrices and quadratic forms. The treatment given here is largely in the spirit of modern linear algebra, and particular attention is paid to the 'geometric' theory of elementary divisors and to decompositions of a vector space into subspaces invariant with respect to a given linear operator. The remaining parts of the book, comprising about eight of the fifteen chapters, are reserved for the investigation of more special topics, of which we mention the following: (i) functions of a matrix (defined in terms of the Lagrange-Sylvester interpolation); (ii) matrix equations; (iii) singular pencils; (iv) non-negative matrices, stochastic matrices, Markov chains; (v) systems of linear differential equations; (vi) the algorithm of Routh-Hurwitz; (vii) Hankel matrices.

Even this brief summary gives some indication of the formidable array of problems deployed with painstaking thoroughness by Professor Gantmacher. Indeed, he does not shrink from lengthy and involved arguments of a kind more commonly found in research papers than in books. However, the material discussed is so interesting and the presentation so clear that the book is sure of a welcome among a wide circle of readers.

L. MIRSKY

Lectures on the Theory of Functions of a Complex Variable. By G. SANSONE and J. GERRETSEN. Vol. 1. Pp. 488. $12.00 1960 (Noordhoff, Groningen)

This very attractive text is based on an earlier Italian language treatise by Sansone. The first volume, on holomorphic functions, discusses Cauchy's theorem, singularities and residues, elliptic functions, the Riemann Zeta function, and summability of power series outside the circle of convergence. Cauchy's theorem is proved for closed curves in an open set of differentiability (Jordan's theorem is not used or proved). Following Artin the theory of singular points is developed without the use of Laurent series. Picard's theorem is proved only for integral functions of finite order, the general case being postponed to the second volume. The book is admirably printed and produced and despite its size, lies flat wherever it is opened.

R. L. G.

Notes on the Quantum Theory of Angular Momentum. By EUGENE FEENBERG and GEORGE EDWARD PAKE. Pp. vii, 56. 10s. 1959. (Stanford University Press)

In this elementary introduction, first published in 1953, some of the most useful formulae for the matrix elements of scalar, vector and tensor operators between angular momentum eigenstates are derived. The results are obtained by elementary (though sometimes rather tedious) calculations, using various commutation rules and recurrence formulae derived from them. No use is made of group theory or Clebsch-Gordan or Racah coefficients. Some applications are given.

A. WEINMANN

Special Functions. By E. D. RAINVILLE. Pp. xii + 365. 82s. 1960. (The Macmillan Company, New York)

This book is based upon lectures given by the author at the University of Michigan. The subject is an enormous one and the author has, of course, had to select fairly drastically. There are two introductory chapters on Infinite Products and Asymptotic Series. It is surprising to find these two topics classed together as "seldom included in standard courses". In old-fashioned courses of function theory, the first appeared more commonly and much earlier than the second, while in a more modern approach there would be some introduction (however naive) to both. In the chapter on asymptotic series there appears to be no very precise definition of what is meant by "$z \to \infty$ in a region R" or, indeed "$z \to c$ in a region R". The vital importance of *how* z tends to infinity or to a finite value in a region is not emphasised. Thus, on p. 34,

$$\lim_{z\to 0 \text{ in } R} \frac{|f(z) - s_n(z)|}{|z|^n} = 0$$

is said to mean that s_n can be made to "approximate $f(z)$ more closely than $|z|^n$ approximates zero by *choosing* z sufficiently close to zero in the region R". Although the book does not take the subject as far as the Stokes phenomenon, the existence of the latter makes the ambiguity implied in "choosing" worth careful attention if nothing else did.

There are chapters on the Gamma, Beta, hypergeometric, generalised hypergeometric, Bessel, confluent hypergeometric, elliptic, theta and Jacobian elliptic functions and on the Legendre, Hermite, Laguerre, Jacobi and ultraspherical and Gegenbauer polynomials. Each topic is discussed in moderate detail and the more important results obtained. Particular stress is laid upon the use of generating functions.

Throughout the book the convenience of the reader is rightly studied rather that that of the printer. The author keeps his notation sufficiently "obvious" to enable the book to be used as a work of reference and there is a useful index. The references and bibliography comfortably avoid the opposite faults of over-elaboration and inadequacy and the book is a welcome addition to the limited range of text-books on the subject.

E. M. WRIGHT

Differential-Gleichungen für Ingenieure: Eine Einführung. By L. Collatz, 2nd Ed. Pp. 197. DM 21.60. 1960. (Teubner, Stuttgart)

Theory and Solution of Ordinary Differential Equations. By D. Greenspan. Pp. viii + 148. 38s. 6d. 1960. (The Macmillan Company, New York)

Of the writing of books on differential equations there is no end, but each of these is a welcome addition to the flow. The first is a thoroughly revised second edition of Professor Collatz's well-known text-book. Into its 197 pages is packed an almost incredible amount of material, accompanied by more than 100 diagrams whose combined clarity and minuteness has its own beauty. The only possible criticism of the format is the narrowness of the margins, which has, at least to this reviewer, a slightly oppressive effect. But this is a small price to pay for getting all this material, carefully and almost leisurely expounded, exemplified and illustrated, in a book which will go into a fair-sized pocket.

The author does not regard "for engineers" as an excuse for vagueness and inaccuracy, but as imposing a need for continual attention to practical examples and to proving results under reasonable (i.e. not pathological) conditions. In addition to the standard topics, there are brief but informative introductions to such topics as non-linear equations.

The second book is essentially a "course" in the subject which "will require approximately *one* semester of intensive study and which will provide the student, no matter what his major interest may be, with some broad insights into this vital subject.

Other objectives which I hope to have incorporated are:

(a) To eliminate the lethargy and mental blocks fostered by the teaching of too many special methods for too many special equations;

(b) To introduce students to other branches of mathematics which are contemporarily playing important roles in differential equations, such as topology;

(c) To introduce students to the very general theories underlying the subject of ordinary differential equations;

(d) To introduce topics of ordinary differential equations which are of basic importance in the applied sciences;

(e) To eliminate the careless habits inculcated by providing answers to exercises which can easily be checked".

Well, no man could hope to do all of that, but Dr. Greenspan has a darned good try. His alternative basing of the existence theorems on the theory of Metric Spaces and Functional Analysis is especially valuable. I wonder why no one ever reproduces Birkhoff and Kellogg's elegant and quite simple 1923 proof of existence theorems of solutions for a wide class of equations. Just as a student could learn best about almost periodic functions from Harold Bohr's three brilliant papers, so perhaps the pioneer application of "fixed point" ideas to existence theorems may be the simplest, as it is the most elegant, introduction to the subject.

E. M. Wright

The Theory of Space, Time and Gravitation. By V. FOCK. Pp. 411. 100s. 1959. (Pergamon Press)

This is a book in which a foremost worker in the field gives a general development of the theory in accordance with his mature conception of it and also a general account of his own major contributions. The general development is an exceedingly illuminating treatment of what are generally called the theories of "special" and "general" relativity, although Professor Fock objects to these terms. The main fundamentals are not different from those of other standard treatments, although there is much individuality in the presentation. However, in some important but less fundamental features, Professor Fock takes a point of view that is more particularly his own. One is in his way of taking care of boundary conditions; another is in his use of a "preferred" (harmonic) coordinate system. While the mathematical convenience of the latter is evident, especially in the later parts of the work, I have not been able to understand the physical significance that Professor Fock seems to attach to it. The later parts are devoted to an account of the derivation of the laws of motion from the law of gravitation, and various problems of approximations in this connexion. Professor Fock has been a pioneer in this development of the theory and all other workers will be delighted to have this survey of his achievements. Professor Kemmer must be congratulated on his service to the subject by his fine translation.

W. H. MCCREA

Mathematische Gesetze der Logik I. By H. A. SCHMIDT. Pp. 555. DM 74.50. 1960. (Springer, Berlin)

In his preface Schmidt says that this book is directed both to mathematicians and non-mathematicians. For the benefit of the former he takes the discussion of each topic up to the central problems and deals with these in a precise mathematical fashion; for the benefit of the latter he carefully develops from first principles all the necessary mathematical background. As a result of this dual aim the book is lengthy; this first volume which deals only with propositional logic runs to 555 pages without being in any sense encyclopaedic. It is clearly intended as a textbook; it does not have the wealth of historical detail and references to recent work which makes Church's 'Introduction to mathematical logic' so valuable. On the other hand it does not have the collection of exercises which makes the latter useful also as a textbook, and it (deliberately) says very little about the philosophical basis of logic. It is undoubtedly useful as a reference book where one can find definitions of all the well known systems of propositional logic and a development of their properties. Despite its length it is a fairly easy book to read because much less is left to the reader than is usual in mathematical texts; the ratio of theorems to meta-theorems is much higher than usual in books on mathematical logic. The reviewer does wonder whether there are many who would have the desire and the patience to acquire such an exhaustive knowledge of propositional logic before passing on to more important

and exciting realms of logic. (A second volume of this work is planned to cover the predicate calculus).

After a few preliminary remarks Schmidt goes on to a fairly full discussion of Boolean algebras and their elementary properties. He then deals with the classical propositional calculus, treating in considerable detail the basic meta-theorems, the truth table method, the functional completeness of various sets of primitives and the use of many valued truth tables in independence proofs. The general notion of formal systems and of the completeness and consistency of formal systems are discussed. Various particular axiom systems for the classical propositional calculus, e.g. Whitehead-Russell, Frege-Lukasiewicz are treated in some detail. The second part of the book deals similarly with non-classical propositional logic, starting with various forms of 'derivative' logic, i.e. systems where implication is treated as a relation of derivability, not as the truth functional material implication. Included are forms of the positive propositional calculus and the 'minimal calculus'. The calculus is finally extended by the addition of the axiom $\neg a \rightarrow \neg\neg a \rightarrow a$ to give the intuitionist propositional calculus. Corresponding sequential calculi of the Gentzen type are next studied. The book ends with two chapters on systems of strict implication of the Lewis type and on modal logic.

J. C. SHEPHERDSON

The Numerical Treatment of Differential Equations. By L. COLLATZ. Third Edition. Pp. 568. DM 98. 1960. (Springer, Berlin)

This English edition of the well known work of Dr. Lothar Collatz is a translation from the second German edition by P. G. Williams, and the opportunity has been taken to make a large number of minor improvements. The general scope of the book is sufficiently indicated by the chapter titles—Mathematical preliminaries and some general principles, Initial-value problems in ordinary differential equations, Boundary-value problems in ordinary differential equations, Boundary-value problems in partial differential equations, Integral and functional equations.

The emphasis is naturally on finite-difference methods and their refinements, including relaxation methods and the method of characteristics, but considerable space is also given to variational methods. A most valuable feature of the treatment is the investigation of error estimation and the development of bracketting theorems (although I regret the absence of any reference to the work of Tosio Kato, J. Phys. Soc. 1949, Japan, 4, 334 and of K.O. Friedrichs, Gött. Nach. 1929, p. 13).

The powerful methods of modern functional analysis are also introduced and there is even a reference to the use of fixed point theorems by Leray and Schrauder in the theory of elliptic differential equations.

The general theory is everywhere abundantly illustrated by fully worked numerical examples and there is a good index.

Dr. Collatz has long been a recognised and outstanding authority on numerical methods and this book is a very welcome exposition of this most important subject.

G. TEMPLE

The Analysis of Variance. By HENRY SCHEFFÉ. Pp. xvi, 477. 112s. 1959. (John Wiley and Sons)

This book is divided into two parts. In the first part are discussed models with fixed effects and independent observations of equal variance. Here the general theory is illustrated by factorial experiments, incomplete designs, and the analysis of covariance. The second part deals with models which call for some degree of randomization, and with the effects of non-normality, inequality of variance and statistical dependence on the models in Part I. There follow mathematical appendices, statistical tables, and comprehensive author and subject indices.

The problems of statistical inference are thoroughly examined, and it would be difficult to find any issues of importance which have been overlooked. Although the layout of the calculations is described, and numerical exercises are included, this is not one of the many books which discuss in detail the presentation and non-mathematical interpretation of experimental results. The author has made many studies on analysis of variance models, and his full survey of their theoretical aspects is a welcome addition to the relevant textbooks.

R. L. PLACKETT

Measurement. By C. W. CHURCHMAN and P. RATOOSH. Pp. 271. 64s. 1959. (Wiley, N.Y. Chapman and Hall, London)

The American Association for Advancement of Science, when it met in 1956, held among its deliberations a symposium on measurement. An endeavour was made to get scientists, for whom measurement meant something different each from the other, to put forward their point of view. The participants were physicists, economists, psychologists and philosophers, with at least one distinguished statistician, E. J. Gumbel. There were sessions on measurement in the physical, social and 'value' sciences, and on formal and general aspects of measurement. The volume under review consists of the papers which were delivered at this symposium.

Given the various interests of the participants it would be a versatile reviewer indeed who could comment usefully on all of the papers. It is, however, fair to say that there is little here of interest mathematically except to the mathematical logician and the natural philosopher. Mathematical statisticians might be interested in "Measurement, Psychophysics and Utility" [S. S. Stevens], "Measurement of Rare Events" [E. J. Gumbel], "Inconsistency of Preferences as a Measure of Psychological Distance" [C. H. Combs], "Experimental Tests of a Stochastic Decision Theory" [D. Davidson and J. Marschak] as also might be experimental psychologists who care to ponder over what they are measuring. Physicists with a philosophical turn of mind might be interested in "Definition and Measurement in Physics" [R. Caws], "Philosophical Problems concerning the measuring of measurement in Physics" [H. Margerau], "Are Physical Magnitudes Operationally Definable" [A. Pap] and "Quantum Theoretical Concept of Measurement" [J. L. McKnight]. The complete strategist is catered for in "A Probabilistic

Theory of Utility and its Relationship to Fechnenian Sealing" [R. D. Luce].

This is a book of essays which those who are interested in pondering over defining what is often undefinable will find both interesting and informative. It is doubtful whether it will have any general appeal.

F. N. DAVID

Mathematical Methods and Theory in Games, Programming, and Economics. 2 volumes. By SAMUEL KARLIN. Pp. 433, 386. 75s. each. 1960. (Pergamon Press)

This work, in two volumes, is concerned with decision processes. The author describes such a process as having typically four parts: a model, a collection of possible decisions, values attached to the several outcomes, and procedures for analysing the effects of decisions. The main component is the model, defined as "a suitable abstraction of reality preserving the essential structure of the problem in such a way that its analysis affords insight into both the original concrete situation and other situations which have the same formal structure". Psychologists, econometricians, political scientists, business men, engineers, in fact everybody constructs such models. The present book deals mainly with cases where the outcome depends not merely on the decision of one single person or team, but also on opponents and on "Nature".

After an Introduction we have Part I: The Theory of Matrix Games, Part II: Linear and Non-Linear Programming and Mathematical Economics, and in volume 2: Theory of Infinite Games.

The treatment is abstract: a game is a triplet $\{X, Y, K\}$ where X and Y are (strategy-) spaces and K is a real-valued function of $x \in X$ and $y \in Y$. Game Theory consists to a large extent in characterising solutions of the functional equation

$$\max_x \min_y K(x, y) = \min_y \max_x K(x, y)$$

sometimes with max and min replaced by sup and inf respectively.

The level of presentation in these volumes is that of the Contributions to the Theory of Games in Annals of Mathematics Studies 24, 28, 39 and 40, to which the author has contributed. Linear Programming is dealt with in a somewhat off-hand way and might have been omitted.

Each chapter includes exercises and historical notes. Solutions or at least hints are given for the exercises, and there is a useful bibliography.

A curious feature of the work should be mentioned. "In order that the [second] volume may be studied independently of volume I, the essential background material from volume I is reproduced here in its entirety, i.e. Chapter 1, which presents the underlying concepts of game theory in their simplest form and introduces the basic notation, and the appendixes which review matrix theory, the properties of convex sets, and miscellaneous topics of function theory."

For advanced game theory and the higher mathematics of econometrics this work is outstanding in its rigour and comprehensiveness. The publishers of this first work in the Addison-Wesley Series in Statistics are to be congratulated on the use of excellent type, print, display of formulae, and paper.

S. VAJDA

Principles of Optics. By MAX BORN and EMIL WOLF. Pp. 803. 120s. 1959. (Pergamon Press)

Max Born's *Optik* was published by Springer in 1933, and, for the last twenty years, his many friends have been urging him to publish an English translation. But when Professor Born came to examine the work which has been published since 1933, he found that the book no longer gave a comprehensive and balanced picture of the field, that a mere translation would be quite inadequate, that a completely new book was essential. This Professor Born has now done, and he was extremely fortunate in being able to secure the help of Dr. Wolf; for no one man could produce alone the massive tome under review.

The sub-title of the book is "Electromagnetic Theory of Propagation, Interference and Diffraction of Light". As this indicates, the field is restricted to those optical phenomena which may be treated in terms of Maxwell's electromagnetic theory of light. No mention is made of the optics of moving media, the optics of X-rays and γ-rays, the theory of spectra, all of which can be treated better in connexion with other fields such as relativity, quantum mechanics, atomic and nuclear physics. Even that important optical instrument, the eye, is dismissed in less than a couple of pages. Within this framework, the authors present a very complete picture of our present knowledge.

The first two chapters deal with the basic properties of the electromagnetic field and the electromagnetic potentials. It is then shown how geometrical optics follows as a limiting case of very short wave-length; the geometrical theory of optical imaging is based on Hamilton's method of characteristic functions, the geometrical theory of aberrations on Schwarzschild's Perturbation Eikonal.

In a long chapter of over 100 pages, the elements of the theory of interference and its applications are discussed. This is followed by 120 pages of elementary diffraction theory based essentially on Kirchhoff's analytical formulation of the Huygens-Fresnel principle.

Up to this point, the book has been concerned mainly with monochromatic light produced by point sources. Chapter X deals with light produced by sources of finite extension and covering a finite frequency range—the subject of partial coherence. The remaining topics dealt with in this book are rigorous diffraction theory, diffraction of light by ultrasonic waves, the optics of metals and crystals.

This is a magnificent book, greatly to the credit of Professor Born, Dr. Wolf and the seven experts who gave them special assistance in certain chapters.

E. T. COPSON

Vector Analysis with Applications to Geometry and Physics. By M. Schwartz, S. Green and W. A. Rutledge. Pp. 556. $8.00. 1960. (Harper and Brothers, New York)

It is perhaps natural that the editor should send this book to a reviewer who has himself written text books covering much the same ground, but the position in which the reviewer finds himself is bound to be a trifle awkward. It is axiomatic that the manner of presentation adopted by the reviewer is that which he, rightly or wrongly, believes to be the best. It is therefore to be expected that he has firm views—his critics might even say rigid views—as to how the subject should be expounded. Consequently, he is unlikely to award high praise to any author whose exposition differs widely from the reviewer's own. On the other hand, if the reviewer is conscious of his own partiality in the matter, he may be reluctant to be forthright in his condemnation of any features of which he may disapprove. In such circumstances it is almost inevitable that his resulting estimate will lie somewhere between faint praise and mild disapproval.

The Preface informs us that this book is primarly for 'the beginning student' but that it can be used either as 'a beginning text' or as a reference for advanced students. The three authors have filled the book's five hundred and fifty pages with a great deal of material most of which is sound, some of which is dull and much of which is useful. A special feature of the book is the very large number of carefully worked examples. Quite a large proportion of the book is devoted to these and for this reason it should be valuable to the weaker student who is unable to solve these examples for himself. The layout of these pages would have been much improved if the printer had used a lead (space) to separate the different examples. The omnibus formulae (p. 150)

$$\oint_\Gamma d\mathbf{r} * \Phi = \int_S (\mathbf{n} \times \nabla) * \Phi \, d\sigma,$$

$$\oint_S \mathbf{n} * \Phi \, d\sigma = \int_V \nabla * \Phi \, d\tau$$

were new to the reviewer and are well worth quoting. In these formulae the symbols $* \Phi$ may be interpreted as the scalar ϕ, as $. \mathbf{a}$, or as $\times \mathbf{a}$; $d\mathbf{r}$ is the vector element of length of the curve Γ, $\mathbf{n}$ is the unit normal to the element $d\sigma$ of the surface S and $d\tau$ is the element of the volume V.

It would be ungracious to catalogue a number of minor blemishes in this text but one major defect must be mentioned. Nowhere in this work are matrices mentioned. This might be understandable in an elementary text, but in one which supposes that the reader is sufficiently mature to cope with the momental ellipse, retarded potentials and elasticity one would have thought that an elementary exposition either of matrices or of tensors would have been more in keeping with modern trends than the fifty odd pages devoted to a not very lucid discussion of dyadics.

D. E. Rutherford

Introduction to the Mechanics of the Solar System. By RUDOLF KURTH. Pp. 177. 42s. 1959. (Pergamon Press)

The four chapters of this book are on the kinematics of a single planet, the dynamics of a single planet, the dynamics of the planetary system, the planets and the moon as rigid bodies. The book is evidently intended for undergraduates doing a first course in celestial mechanics, and the author has himself used the material in this way. On the other hand, the reviewer has never given such a course and he can express only an *a priori* opinion. Unlike Dr. Kurth, he would not think the subject a good one in which to give special attention to the inferring of the laws of motion and of gravitation. For a formulation even as careful as Dr. Kurth's begs a number of questions about time and space. Thus the reviewer would expect to have to choose outright between either a course on the relevant parts of scientific inference, which would scarcely come properly under the title of the book, or else a course on celestial mechanics treated as a special technical application of known classical mechanics. On this latter aspect, Dr. Kurth has some very interesting things to say, particularly on the treatment of perturbation theory by Poisson's method of varying the parameters in the first integrals of the unperturbed motion rather than in the solution-functions. Indeed, if the reviewer had to give a course on any aspect of the subject he would gladly and gratefully take a number of ideas from this book.

W. H. MCCREA

Les Systèmes d'idéaux. By P. JAFFARD. Pp. 132. NF 25. 1960. (Dunod, Paris)

Multiplicative ideal theory has passed through several stages of abstraction and the form given to it by Lorenzen in 1939 may well be considered definitive. While the ultimate aim is the study of divisibility properties in integral domains, the starting point is the notion of a partially ordered abelian group. In any commutative ring the principal ideals, in contrast to the general ideals, may be defined in terms of the multiplication alone. Accordingly an ideal system in an ordered abelian group is a closure system of sets which includes all principal ideals and admits multiplication by elements. The ideals in a ring form a particular ideal system, which however shares many properties with general ideal systems.

The author's aim has been to give an account of this development which presupposes only an elementary knowledge of groups and rings. Within the limitation to the commutative case he treats partially ordered groups and their representation (when possible) as subdirect products of totally ordered groups. The applications to rings include 'permanence' theorems showing what properties are transmitted to rings of fractions etc. and other results such as the 'going down' theorem and the extension of specializations. The proofs are commendably brief, although one has the impression that this is not because they are developed in this general context. The approach does however have the

advantage of bringing some order into a bewildering variety of rings (more than twelve different types!) and there are two diagrams showing some of their interrelations. The book also has a substantial bibliography and index.

The author has achieved a nice balance in his degree of generality; this should make the book both pleasant to read as an introduction and useful as a reference book.

P. M. COHN

BRIEF MENTION

Das Kontinuum und Andere Monographien. By H. WEYL. $6.00. 1960. (Chelsea, New York)

This volume brings together Weyl's celebrated foundations' study of 1917, his lectures on Geometry (axiomatic and differential geometry) given in Spain in 1922, and his study of Riemann's dissertation on the hypotheses of the foundations of geometry. The fourth monograph is a work of Landau's on Complex Function Theory (containing his celebrated proofs of Picard's theorem).

Combinatory Analysis. By P. A. MACMAHON. Vols. I, II bound together. Pp. 302, 340. $7.50. 1960. (Chelsea, New York)

An unaltered reprint of the 1915, 1916 edition by the Cambridge University Press.

Asymptotic Series and Divergent Series. By W. FORD. Pp. 342. $6.00. 1960. (Chelsea, New York)

This volume brings together Ford's *Studies on Divergent Series and Summability*, published in 1916, and *The Asymptotic Developments of Functions Defined by MacLaurin Series*, published in 1936.

Statics. By H. LAMB. Pp. 357. 18s. 6d. 1961. (Cambridge University Press)

This students' edition is the eighth reprint of the third edition of 1933.

A Course of Analysis. By E. G. PHILLIPS. Pp. 361. 15s. 1961. (Cambridge University Press)

This students' edition is the sixth reprint of the 1939 second edition.

Special Functions of Mathematical Physics and Chemistry. By I. N. SNEDDON. Second Edition. Pp. 184. 10s. 6d. 1961. (Oliver and Boyd)

The first edition (1956) was reviewed by T. M. MacRobert, *Gazette* XLI, p. 222.

A Treatise on the Differential Geometry of Curves and Surfaces. By L. P. EISENHART. Pp. 474. $2.75. 1960.

Advanced Euclidean Geometry. By R. A. JOHNSON. Pp. 319. $1.65. 1960.

Elements of Projective Geometry. By LUIGI CREMONA. Pp. 302. $1.75. 1960.

Three more welcome additions to the Dover reprints. Two are very well-known; Johnson's book deals with such topics as Brocard points, Tucker circles, the Steiner point of a triangle and the Neuberg circles.

Mathematical Puzzles and Diversions from Scientific American. By MARTIN GARDNER. Pp. 163. 17s. 6d. 1961. (Bell and Sons, London)

Amongst the puzzles and diversions featured are Hexaflexagrams, noughts and crosses, turning an inner tube inside out through a hole in its side, polyominoes and a new variation on the counterfeit coin problem, much simpler than the original but most instructive. This British edition of an American book has been reset and printed here.

THE MATHEMATICAL ASSOCIATION

The fundamental aim of the Mathematical Association is to promote good methods of Mathematical teaching. Intending members of the Association are requested to communicate with one of the Secretaries. The subscription to the Association is 21s. per annum and is due on January 1st. Each member receives a copy of the *Mathematical Gazette* and a copy of each new Report as it is issued.

Change of address should be notified to the Membership Secretary, Mr. R. E. Green. If copies of the *Gazette* fail to reach a member for lack of such notification, duplicate copies can be supplied only at the published price. If change of address is the result of a change of appointment, the Membership Secretary will be glad to be informed.

Subscriptions should be paid to the Hon. Treasurer of the Mathematical Association.

The **Library** of the Mathematical Association is housed in the University Library, Leicester.

The address of the Association and of the Hon. Treasurer and Secretaries is **Gordon House, 29 Gordon Square, London, W.C.1.**

BRANCH REPORTS

THE CARDIFF BRANCH

Report for 1960–61

Chairman: Dr. A. C. Bassett.
Treasurer: Mr. R. A. Jones.
Secretary: Mr. W. H. Williams.

Monday, October 17th, 1960: Mr. I. E. Hughes H.M.I., the retiring chairman, gave an account of "Some Interesting Problems".

Monday, November 21st, 1960: Mr. Hubert Phillips gave a talk on "The Laws of Chance".

Monday, January 23rd, 1961: Professor P. T. Landsberg gave an interesting address on "Information Theory".

Monday, March 6th, 1961: Mr. D. T. Steer of the Steel Company of Wales gave a talk on "The Simulation Technique for Industrial Problems".

In conjunction with the Mathematics Department of the Welsh College of Advanced Technology, the Branch participated in a programme for Commonwealth Technical Training Week involving a lecture on "Electronic Digital Computers" and visits of school parties to the Stantec Zebra Computer at the College.

W. H. Williams (Hon. Sec.)

NOTTINGHAM AND DISTRICT BRANCH

Report for the year 1959/60

The Annual General Meeting was held on the 5th December at the Institute of Education, Nottingham University. After the official business, Dr. D. Thompson and Mr. N. Worskett of the Willenhall Comprehensive Secondary School spoke on the topic of 'Mathematics in the Comprehensive School'. A survey of the type of work involved, the examinations for which certain pupils were prepared, the methods of grouping and the teaching aids used, were all discussed. The large number of questions put to the speakers was evidence of the interest they had created.

On Friday, 26th February, the Branch held its first Mathematical Quiz, with an attendance of more than 200, when a team of pupils from Nottingham City Grammar School competed against a team from the County Grammar Schools.

The Spring meeting took the form of a one day conference held at Nottingham University on March 12th, when the main theme was

'How much School Mathematics?'. Dr. M. Jackson of Nottingham University spoke on behalf of the Universities, Mr. A. J. L. Avery of Derby and District College of Technology gave the views of the technical colleges and Mr. R. Illing of Kesteven Training College surveyed the needs of the training colleges. For the first part of the afternoon session, Mr. C. T. Daltry of the Institute of Education, London University took the chair for a discussion of the morning topic. A lively discussion followed which completed an interesting survey of the topic. The final speaker, Dr. G. C. Shephard of Birmingham University gave a very instructive and entertaining lecture on 'Polyhedra', in which he demonstrated a large number of models. These models created a great deal of interest among the members of the audience.

On June 18th the Summer meeting was held at the Institute of Education, Nottingham University when the two speakers were members of the Leicester Branch. Mr. J. W. Hesselgreaves spoke on 'Probability–or–Watch how you go', when several topical problems in probability were discussed. This was an illuminating and interesting address. Mr. K. F. Solloway followed with the topic 'See for yourself' in which he introduced some methods and apparatus which he had devised to help achieve sight for the mathematically blind.

The Officers for the year were:

President Dr. G. Power. Vice-President Mr. K. R. Imeson.

Treasurer Mr. C. R. Swaby. Secretary Mr. H. L. W. Jackson.

IPSWICH AND DISTRICT BRANCH

Report for 1960–1961

The Branch was formed in March 1960 and now has over 50 members, of whom about 20 are members of the Association. The following meetings have been held at various schools and the average attendance was nearly forty.

16th March 1960. Inaugural meeting. Mr. A. P. Rollett spoke on "Mathematical Gastronomy".

30th May, 1960. "Errors and Omissions not Excepted" by Dr. E. A. Maxwell.

26th October, 1960. Mr. B. J. F. Dorrington outlined the new Diploma in Mathematics.

5th December, 1960. "Mathematics in Agriculture" by Mr. J. C. Dickens.

25th February, 1961. A joint meeting with the East Suffolk Group of the A.T.A.M. Miss Giuseppi spoke on "Mathematics in Secondary Modern Schools" and there was an exhibition of teaching aids.

3rd May, 1961. "The Training of Mathematics Teachers" by Mrs. E. M. Williams.
6th June, 1961. "Some Fun with Probability" by Mr. W. Hope-Jones. A large number of school pupils attended and enjoyed this meeting.

At the first A.G.M. the following were elected: President, Mr. A. Walker; Secretary, Mr. W. A. Dodd; Treasurer, Mr. K. J. Playforth; Committee, Mrs. B. Parker, Mr. T. C. Grice, Mr. D. Leighton.

W. A. Dodd, Hon. Sec.

NORTH-EASTERN BRANCH

Report for the Session 1960–61

The following meetings were held:

26th October, 1960
Mr. C. A. Stewart of Rutherford College of Technology gave a talk (accompanied by demonstrations) on "Analogue Computers".

16th November, 1960
Dr. N. du Plessis of King's College, spoke on "Structure in Mathematics".

10th December, 1960
Professor G. S. Rushbrooke, of King's College, addressed the Branch on "Experimental Mathematics".

28th January, 1961
Dr. S. Holgate, Master of Grey College in the University of Durham, gave an address entitled "Alternatively".

8th February, 1961
Professor G. E. H. Reuter of Durham gave the Branch Annual Lecture to sixth form pupils on "Random Walks".

25th February, 1961
This meeting was held in conjunction with the University of Durham School Examination Board Annual Conference. Mr. R. Wooldridge of the Lanchester College of Technology spoke on "Computers and Schools".

14th March, 1961
Mr. B. J. F. Dorrington visited the Branch to speak on the subject of "The Diploma in Mathematics".

Officers of the Branch for 1961–63 are:

President: Professor G. E. H. Reuter.
Hon. Sec.: Mr. J. F. Reed.
Hon. Treas.: Miss J. A. Scott.

J. F. Reed, Hon. Sec.

QUEENSLAND BRANCH

Annual Report for the year 1960–1961.

At the Annual Meeting held on 5th April, 1960, the following officers were elected for 1960—President: Professor C. S. Davis. Vice-Presidents: Messrs. H. M. Finucan, A. W. Young. Hon. Secretary-Treasurer: Mr. M. P. O'Donnell. Members of Committee: Miss M. A. Maclean, Messrs. G. T. Evans, G. L. Hubbard, Associate Professor J. P. McCarthy, Mr. P. B. McGovern.

The Presidential Address, entitled "Inequalities Associated with a Triangle" was given by the retiring President, Professor C. S. Davis.

In 1960, the Queensland State Government formed a Committee to inquire into Secondary Education. The Director-General of Education invited the Branch to make a submission. The Branch appointed a Committee, with Associate Professor J. P. McCarthy as convener, to draft a submission, and a meeting on 28th June was devoted to a discussion of the draft. The Branch's submission, sent to the Government Committee on 26th July, embodied suggestions about the length of secondary education, public examinations, and the various mathematics courses which should be offered in schools.

Five general meetings were held during the year as follows:

16th May, 1960:	Papers by Mr. C. W. Radcliffe and Associate Professor J. P. McCarthy on "The Junior Arithmetic Syllabus".
28th June, 1960:	Discussion of the submission to the Committee to inquire into Secondary Education.
26th July, 1960:	Paper by Dr. E. T. Steller on "Actuarial Mathematics and Human Reactions to Collective Treatment".
28th Sept., 1960:	Papers by Messrs. H. M. Finucan and A. W. Young on "Co-ordinate Geometry, and its Introduction into the Senior Mathematics I Syllabus".
2nd Nov., 1960:	Paper by Dr. C. Gattegno on "A Summary of a Mathematical Tour of Australia".

Mr. E. W. Jones, a Life Member of the Mathematical Association, died during the year. Mr. Jones gave long and valued service to the Branch as committee member (1926–1950), vice-president (1950–1956), and president (1956–1958); during these years Mr. Jones read 20 papers at meetings.

Membership of the Branch is at present 66, of whom 28 are ordinary members of the Mathematical Association. At the meetings

held during 1960, 15 new members were admitted. The average attendance at meetings was 30. Copies of the *Gazette* are circulated among associate members.

M. P. O'DONNELL,
Hon. Secretary.

Queensland Branch of the Mathematical Association.

THE LIVERPOOL MATHEMATICAL SOCIETY

Report for 1960–1961

1960

Monday October 17	CLOCKS AND RELATIVITY Professor J. L. Synge, F.R.S. (Dublin Institute of Advanced Studies)
Monday November 21	ON PLANE POLYGONS Dr. B. H. Neumann, F.R.S. (Manchester)
Monday December 5	TRENDS IN SECONDARY MATHEMATICS An account of some international developments. Mr. C. Hope (Worcester)

1961

Monday January 16	A Society discussion meeting will be opened by Sir James Mountford, the Vice-Chancellor of the University of Liverpool, who will speak on "Some problems of University entrance requirements"
Monday March 6	HIGH ALTITUDE RESEARCH WITH THE SKYLARK ROCKET Dr. J. B. Dorling (Farnborough)
Monday May 8	(i) Annual General Meeting (ii) President's address

The January meeting will be held in the Large Lecture Theatre in the New Medical School, Ashton Street, from 5.00 p.m. to 6.00 p.m. All other meetings will be in the Mathematical Institute of Liverpool University, Elizabeth Street, Brownlow Hill, at 5.00 p.m. (Tea at 4.30 p.m.).

The annual subscription is 4/6d and should be paid to the Treasurer (Dr. W. F. Newns, Mathematical Institute, The University, Liverpool 3).

Membership is open to all interested in Mathematics, and the

Society invites anyone interested to come as a guest to the first meeting on October 17th.

G. R. BALDOCK,
(President),
Mathematical Institute,
The University,
Liverpool 3.

M. C. R. BUTLER,
(Secretary),
Mathematical Institute,
The University,
Liverpool 3.

THE MATHEMATICAL ASSOCIATION

REPORT OF THE COUNCIL FOR THE YEAR 1960

Membership

During the year ended 31 October, 1960, 515 ordinary Members and 181 Junior Members joined the Association. Final membership figures were: Honorary 8; Ordinary 3700; Junior 413; Life 264. The total, 4385, shows an increase of 531 during the year. This is almost the same increase as in 1959, but includes 435 Ordinary Members as against 337 then.

The Council regrets to report the death of W. F. Beard (1898), Miss W. E. Bennett (1937), Miss A. M. Bozman (1926), L. Clitheroe (1936), Mrs. L. J. Collins (1923), Brychan Davies (1921), Dr. J. Dougall (1928), H. J. Edwards (1930), R.-V. Goormaghtigh (1947), P. M. Grundy (1942), J. W. Lewis (1913), Prof. Emeritus V. S. Mallory (1931), C. D. T. Owen (1935), B. R. Shorthouse (1957), Miss H. Wenham (1908), and Prof. H. A. Zager (1938).

Finance

The Treasurer's statement for the year ended 31 October, 1960 shows an excess of income over expenditure of £3200.6.1.

The total income of £7951.5.0. includes £3530.12.2. from subscriptions, an increase of over £500 on the previous year, and the highest in the history of the Association. Reports, particularly those on the teaching of Mathematics in Primary Schools and Secondary Modern Schools, have sold well, and during the year a total of £2883.11.1. has been received from this source. Nearly all the covenanted subscriptions have now expired and only £66.14.0. was recovered under this heading. Members are reminded that while it is possible for them to claim personal tax relief on the amount of their subscriptions, it is also still open to them to covenant their subscriptions to the Association, which benefits substantially.

Expenditure during the year amounted to £4751.5.0., but owing to the late arrival of the bill for the October *Gazette* (some £750), the accounts include the cost of only three *Gazettes* instead of the usual four. Three reports were reprinted at a cost of £783.3.5., but

the heavy outlay incurred last year on the printing of new reports was not repeated. Considerable expenditure of this nature is expected in the next two years, but though this is likely to be heavy, the resources of the Association are now adequate to meet these demands.

The Mathematical Gazette

The circulation of the *Gazette* has continued to grow and 6000 copies of the December issue were printed. The increased demand for advertising space noted last year has been maintained.

The response to the request for general articles of an elementary nature has been most encouraging. The flow of suitable material for 'Mathematical Notes' has become a flood and now greatly exceeds the space available, making considerable delay in publication inevitable. Members should not be discouraged by this; their contributions will be welcomed. Reviews, except elementary reviews which receive priority, are also subject to delays, amounting often to 18 months or more.

Library

Periodicals continue to constitute the Library's chief accessions. A start has been made in preparing a printed list of books in the Library to facilitate borrowing. The Librarian is anxious to build up a complete set of *Gazettes* and gifts of early numbers would be warmly welcomed.

Teaching Committee

Much hard work has been done by several sub-committees. The work of the Analysis and the Statistics Sub-Committees has reached an advanced stage; it has been decided to prepare a new report on Mechanics, re-issue of the existing report in revised form having proved impracticable. In addition to the regular sub-committees, three special panels are preparing memoranda for presentation to the Stockholm International Conference of Mathematicians in 1962, on (i) modern mathematics in schools, (ii) the training of mathematics teachers, (iii) the teaching of arithmetic and algebra to children to age 15.

Sales of the Association's reports continue at a good level.

Mr. Dorrington having resigned the secretaryship on taking up another office in the Association, the Teaching Committee wishes to record its appreciation of his work during the past 7 years.

Diploma in Mathematics

Regulations for the Diploma have been circulated to all members of the Association, to all Training Colleges, Departments of Education, Local Education Authorities, etc. The examiners have produced specimen papers.

The Council has nominated the following members to constitute the Examinations Board:

Dr. I. W. Busbridge, Mr. J. T. Combridge, Mr. B. J. F. Dorrington, Mr. W. J. Langford and Dr. E. A. Maxwell.

The Branches

The list of branches of the Association continues to grow. Two have been formed in this country during the past year, one at Ipswich and one at Wolverhampton. A new branch has also been formed at Christchurch, New Zealand.

News received from the branches indicates that they flourish and that their activities are expanding. A particularly encouraging sign is that more branches are arranging meetings of especial interest to members of sixth forms in grammar schools.

Problem Bureau

The Bureau has again been active, especially for a month or two before the University Entrance Scholarship examinations. In a few cases there have been requests for a number of solutions at the same time, with consequent delay in replying. Not more than six solutions should be asked for at a time. It is nine years since the names of the "staff" (the inverted commas indicate that the rewards of service are not of a material kind) were given. The current list is R. H. Cobb, W. J. Hodgetts, W. Hope-Jones, R. V. H. Roseveare, D. A. Young, C. A. Ford, A. T. F. Nice, R. W. Payne and H. G. Woyda. Others are referred to occasionally as specialists, but the above assist regularly; their skill and tenacity cannot be too highly praised.

As before, a copy of the questions and a stamped addressed envelope should be sent. In the case of Cambridge Entrance Scholarship examinations in Mathematics a reference, such as 294/27/9, to the published books of papers is enough. All correspondence should be addressed to Dr. G. A. Garreau, 90 Wyatt Park Road, London S.W.2.

Officers & Council

Council wishes to record its thanks to the President, Dr. E. A. Maxwell, and to the Officers, for their efforts on behalf of members. The task of conducting the affairs of the Association continues to grow. A change of rules to be proposed at the Annual General Meeting in April 1961 will, if agreed, allow for the appointment of Assistant Secretaries.

COUNCIL—1961

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J. T. COMBRIDGE, M.A., M.Sc.

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Miss W. A. COOKE, B.Sc., 6 Cherry Orchard, Stoke Poges, Bucks.

Honorary Assistant Secretaries:

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R. E. GREEN, M.A.
Miss R. K. TOBIAS, M.A.

Honorary Librarian and Editor of "The Mathematical Gazette".

Professor R. L. GOODSTEIN, M.Sc., Ph.D., D.Litt., University, Leicester.

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Secretary: T. R. THEAKSTON, B.Sc., Charter Avenue, Coventry, Warks.

Representatives of Branches Committee:

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Secretary: W. E. REMINGTON, B.Sc., 2 Somerville Road, Leicester.

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G. POWER, M.B.E. M.A., Ph.D., Nottingham.
F. J. HAWLEY, M.Sc., Wolverhampton.

Other Members of the Council for 1961:

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BRANCH SECRETARIES

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NORTH STAFFS.:	E. R. KILFOIL, 92 Baddeley Green Lane, Milton, Stoke-on-Trent, Staffs.
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EXETER:	Miss N. A. COMERFORD, 5 Broad Parks Avenue, Exeter, Devon.
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OXFORD:	F. A. GARSIDE, Magdalen College School, Oxford.
IPSWICH AND DISTRICT:	W. A. DODD, 34 Dorset Close, Ipswich, Suffolk.
MALAYA:	R. K. GUY, University of Malaya, Cluny Road, Singapore 10.
SIERRA LEONE:	N. A. HAYES, Collegiate School, Freetown, Sierra Leone.
NEW SOUTH WALES:	A. R. BUNKER, The Teachers' College, Sydney.
QUEENSLAND:	M. P. O'DONNELL, Department of Mathematics, The University of Queensland, Brisbane.
VICTORIA:	G. R. CLARK, 8 Jordan Grove, Glen Waverley, Victoria.
AUCKLAND, NEW ZEALAND:	E. H. DRIVER, Auckland Grammar School, Auckland.
WELLINGTON, NEW ZEALAND:	H. M. SMYTH, Wellington College, Wellington, C.4.
CHRISTCHURCH, NEW ZEALAND:	R. HARRIES, Linwood High School, Aldwins Road, Linwood, Christchurch, N.Z.
WESTERN AUSTRALIA:	DR. H. BRINER, Mathematics Department, University of Western Australia, Nedlands.
SOUTHERN AUSTRALIA:	I. P. LANG, Woodville High School, Adelaide.

Annual General Meeting 1962 will be held at King's College, London, April 17th and 18th.

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Just Published

HOMOGENEOUS COORDINATES

by C. V. DURELL, M.A. 212 *pages.* 14*s.* *Key,* 6*s. net.*

This book is a sequel to the author's recent and highly praised *Elementary Coordinate Geometry*. It bridges the gap between G.C.E. scholarship level and the requirements for mathematical scholarships at the Universities.

The range of the book is much the same as that of the author's *Algebraic Geometry* but the subject-matter is presented here in a *simplified* and *far less detailed* form. *Brief Contents:* Point-coordinates in Cartesian Geometry. Line-coordinates in Cartesian Geometry. The Projective Transformation. Duality. Homography and Involution. Theorems of Chasles and Pascal. Homography and Involution on a Conic. Desargues' Theorem. Triangle and Conic. Reciprocation. Projective and Cartesian Geometry.

MATHEMATICS IN YOUR WORLD

by KARL MENNINGER. *Illustrated.* 21*s. net.*

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